Stability of travelling-wave solutions for reaction-diffusion-convection systems

E.C.M.Crooks

Abstract

We are concerned with the asymptotic behaviour of classical solutions of systems of the form

$$(1)\qquad \begin{cases} u_t = Au_{xx} + f(u, u_x), & x \in \mathbb{R}, t > 0, u(x,t) \in \mathbb{R}^N, \\ u(x,0) = \phi(x), \end{cases}$$

where A is a positive-definite diagonal matrix and f is a "bistable" nonlinearity satisfying conditions which guarantee the existence of a comparison principle for (1). Suppose that (1) has a travelling-front solution w with velocity c, that connects two stable equilibria of f. (There are hypotheses on f under which such a front is known to exist [5].) We show that if ϕ is bounded, uniformly continuously differentiable and such that $\|w(x) - \phi(x)\|$ is small when $|x|$ is large, then there exists $\chi \in \mathbb{R}$ such that

$$(2)\qquad \|u(\cdot, t) - w(\cdot + \chi - ct)\|_{BUC^1} \to 0 \quad \text{as} \quad t \to \infty.$$

Our approach extends an idea developed by Roquejoffre, Terman and Volpert in the convectionless case, where f is independent of u_x. First ϕ is assumed to be increasing in x, and (2) proved via a homotopy argument. Then we deduce the result for arbitrary ϕ by showing that there is an increasing function in the $\omega-$limit set of ϕ.

1 Introduction

This paper is concerned with the asymptotic behaviour of classical solutions of the system

$$(3)\qquad u_t = Au_{xx} + f(u, u_x), \quad x \in \mathbb{R},\ t > 0,\ u(x,t) \in \mathbb{R}^N,$$

$$(4)\qquad u(x,0) = \phi(x), \quad x \in \mathbb{R},$$

under the following hypotheses:

(a) A is a positive-definite diagonal $N \times N$ matrix,

$f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuously-differentiable function such that

(f1) $f_i(q,p) = \tilde{f}_i(q_1, \ldots, q_N, p_i)$ (the i-th component of f does not depend on p_j for $j \neq i$),

(f2) $\dfrac{\partial f_i}{\partial q_j}(q,p) > 0, \quad i \neq j,\ i,j = 1, \ldots, N, \quad (q,p) \in \mathbb{R}^N \times \mathbb{R}^N,$

(f3) $f(E^-,0) = f(E^+,0) = 0$, where $E^- < E^+$, $E^\pm \in \mathbb{R}^N$ and all the eigenvalues of $d_q f[E^\pm, 0]$ lie in the open left-half complex plane (*bistability condition*),

(f4) there exists $\gamma \in (1,2)$ and an increasing function $\mu : [0,\infty) \to [0,\infty)$ such that for each $p, q \in \mathbb{R}^N$,

$$\|f(q,p)\| \le \mu(\|q\|)(1 + \|p\|^\gamma) \quad (\|\cdot\| \text{ denotes a norm on } \mathbb{R}^N)$$

and

(TW) there exists a monotone travelling-wave solution $w(x-ct)$ of (3) such that $w(x) \to E^\pm$ as $x \to \pm\infty$, and $w'(x) > 0$ is bounded independently of x. (In fact, these properties of w together with the above hypotheses on f ensure that $w'(x) \to 0$ at an exponential rate as $|x| \to \infty$. See the remark following the proof of Lemma 2.5.)

Note that [5] proves the existence of a wave w satisfying **(TW)** under hypotheses similar, though not identical, to **(a), (f1)-(f4)**, together with an assumption on the nonexistence of stable equilibria of f between $(E^-,0)$ and $(E^+,0)$. Such equilibria could prevent the existence of a front connecting E^- to E^+ - see [7]. For the *scalar* bistable *equation* (3), in the convectionless case when $f \in \mathbb{R}$ and is independent of u_x, convergence to a travelling-front solution w from initial data ϕ is comprehensively treated in [7]. Stability of fronts for bistable convectionless *systems* is developed in [14] and [13]. Here we extend this work to nonlinearities dependent on u_x.

Throughout, $\mathfrak{e} = (1,\dots,1)$ and $d_q f[q,p]$, $d_p f[q,p]$ denote the partial Fréchet derivatives of f at $(q,p) \in \mathbb{R}^N \times \mathbb{R}^N$ with respect to the first and second arguments of f respectively. If $q^\pm \in \mathbb{R}^N$, then $q^- < (\le) q^+$ if $q_i^- < (\le) q_i^+$ for each $i \in \{1,\dots,N\}$; $[q^-, q^+]$ denotes the set of $q \in \mathbb{R}^N$ such that $q^- \le q \le q^+$. For Υ a subset of a real or complex vector space, $k \in \mathbb{N} \cup \{\infty\}$, $\mathfrak{C}^k(\mathbb{R},\Upsilon) = BUC^k(\mathbb{R},\Upsilon)$, the space of functions $g : \mathbb{R} \to \Upsilon$ such that g and the derivatives of g of order less than or equal to k are bounded and uniformly continuous on $\mathbb{R}$. For brevity, we write $\mathfrak{C}^k = \mathfrak{C}^k(\mathbb{R},\mathbb{R}^N)$ and $\widetilde{\mathfrak{C}}^k = \mathfrak{C}^k(\mathbb{R},\mathbb{C}^N)$.

Known results yield, under hypotheses **(a)**, **(f1) - (f4)**, that there exists $\epsilon > 0$ such that system (3 - 4) with initial data $\phi \in \mathfrak{C}^1(\mathbb{R}, [E^- - \epsilon\mathfrak{e}, E^+ + \epsilon\mathfrak{e}])$ has a unique classical solution u^ϕ that exists for all time and depends continuously in $\mathfrak{C}^1$ on the initial data ϕ. See the Appendix for references. We will prove that if $\phi \in \mathfrak{C}^1$ is such that $\|w(x) - \phi(x)\|$ is small when $|x|$ is large, then u^ϕ converges to a shift of the travelling wave w, in the sense that there exists $\chi \in \mathbb{R}$, depending on ϕ, such that

$$\|u^\phi(\cdot,t) - w(\cdot + \chi - ct)\|_{\mathfrak{C}^1} \to 0 \quad \text{as} \quad t \to \infty. \tag{5}$$

Let $v(x,t) = u(x+ct,t)$, where u is a solution of (3). Then

$$v_t = Av_{xx} + cv_x + f(v, v_x). \tag{6}$$

Note that w is a stationary solution of (6) and that $v(x,0) = u(x,0)$ for all $x \in \mathbb{R}$. We seek $\chi \in \mathbb{R}$ such that

$$\|v^\phi(\cdot,t) - w(\cdot + \chi)\|_{\mathfrak{C}^1} \to 0 \quad \text{as} \quad t \to \infty. \tag{7}$$

(v^ϕ will denote the unique classical solution of (6) with initial data $\phi \in \mathfrak{C}^1(\mathbb{R}, [E^- - \epsilon\mathfrak{e}, E^+ + \epsilon\mathfrak{e}])$ throughout.)

To prove (7), it will first be shown, in Theorem 3.1, that w is “locally”stable in $\mathfrak{C}^1$; that is, given initial data ϕ which is a sufficiently small $\mathfrak{C}^1$-perturbation of w, the corresponding solution v^ϕ of (6) converges in $\mathfrak{C}^1$ to a translate of w as $t \to \infty$. This is a consequence of the fact that the

spectrum of the linearisation of (6) about w is in a sector in the open left-half plane, except for a simple eigenvalue at zero caused by the translation invariance of (6). For $g \in \mathfrak{C}^2$ define

$$\begin{aligned} \mathcal{L}g(x) &= Ag''(x) + \{c + d_p f[w(x), w'(x)]\}g'(x) + d_q f[w(x), w'(x)]g(x) \\ &= Ag''(x) + C(x)g'(x) + B(x)g(x), \end{aligned} \tag{8}$$

say; $B, C : \mathbb{R} \to M^{N\times N}$ are uniformly continuous $N \times N$-matrix-valued functions of x. Consider $\mathcal{L}$ as an operator acting in $\mathfrak{C}$, with domain $\mathfrak{C}^2$. We abuse notation slightly by also using the symbol $\mathcal{L}$ for the complexification of $\mathcal{L}$ when appropriate. The spectrum of $\mathcal{L}$ is analysed in section 2. Section 3 is devoted to proving local stability of w in $\mathfrak{C}^1$, following a method in [8].

The main convergence result, Theorem 5.4, is proved in two steps. First, in section 4, $\phi \in \mathfrak{C}^1$ is assumed to be increasing, and convergent to $E^{\pm}$ at $\pm\infty$ respectively. Our approach derives from that of [14]. A function ϕ^* is constructed from ϕ and the wave w so that the solution v^{ϕ^*} of (6) corresponding to initial data ϕ^* satisfies (7). The corresponding result for v^{ϕ} is then deduced using a homotopy argument. Section 5 concludes the paper by showing that for more general initial data ϕ, close to w at infinity, there is an increasing function in the ω-limit set of ϕ. This last step is motivated by [13]. Note that the main convergence theorem Theorem 5.4 implies uniqueness of travelling-front solutions of (3) within a certain class - see Corollary 5.5 for details.

In an Appendix, we state some useful known results for (6) - namely a comparison principle, local/global existence theorems and *a priori* bounds. Some wave-dependent sub- and super-solutions, useful in the stability analysis of w, are also given. This material will often be referred to in the body of the paper.

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2 Properties of $\mathcal{L}$

Let Y, W be complex Banach spaces and let $L(Y, W)$ denote the space of bounded linear operators from Y into W. A linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset Y \to Y$ is said to be *sectorial in* Y if it is a closed densely-defined operator such that for some $\omega \in \mathbb{R}, \theta \in (\frac{\pi}{2}, \pi), M > 0$,

$$\Sigma = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\} \subset \rho(\mathcal{A}), \quad \text{the resolvent set of } \mathcal{A},$$

and

$$\|(\lambda I - \mathcal{A})^{-1}\|_{L(Y,Y)} \leq \frac{M}{|\lambda - \omega|} \quad \text{for all } \lambda \in \Sigma,$$

(see [11, p 33]). If $\mathcal{A}$ is sectorial in Y, then $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup $e^{t\mathcal{A}}$ in the Banach space Y.

Lemma 2.1 *The operator $\mathcal{L} : \mathfrak{C}^2 \subset \mathfrak{C} \to \mathfrak{C}$ defined in (8) is sectorial in $\mathfrak{C}$.*

Proof. In (8), the matrices A and $C(\cdot)$ are diagonal. It follows from the scalar-valued-equation analysis of [11, p 81, Corollary 3.1.9 (ii)] that the operator $\mathcal{T} : \mathfrak{C}^2 \subset \mathfrak{C} \to \mathfrak{C}$ defined by $\mathcal{T}g = Ag'' + C(\cdot)g'$ is sectorial.

Define $\mathcal{S} : \mathfrak{C} \to \mathfrak{C}$ by $\mathcal{S}g = B(\cdot)g$. Clearly $\mathcal{S} \in L(\mathfrak{C}, \mathfrak{C})$. So [11, p 64, Proposition 2.4.1] yields that $\mathcal{L} = \mathcal{T} + \mathcal{S}, \mathcal{L} : \mathfrak{C}^2 \subset \mathfrak{C} \to \mathfrak{C}$ is sectorial. $\square$

For $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset Y \to Y$ and $Y^\sharp$ be a Banach space with $\mathcal{D}(\mathcal{A}) \subset Y^\sharp$ and $Y^\sharp \hookrightarrow Y$, where $\hookrightarrow$ denotes continuous embedding, let the *part* of $\mathcal{A}$ in $Y^\sharp$[11, p 40] be $\mathcal{A}^\sharp$, where

$$\mathcal{D}(\mathcal{A}^\sharp) = \{g \in \mathcal{D}(\mathcal{A}) : \mathcal{A}g \in Y^\sharp\} \subset Y^\sharp, \quad \text{and} \quad \mathcal{A}^\sharp g = \mathcal{A}g \quad \text{for each } g \in \mathcal{D}(\mathcal{A}^\sharp).$$

Lemma 2.2 *The part of $\mathcal{L}$ in $\mathfrak{C}^1$ is sectorial in $\mathfrak{C}^1$.*

Proof. Define $\mathcal{M} : \mathfrak{C}^2 \subset \mathfrak{C} \to \mathfrak{C}$ by $\mathcal{M}g = Ag''$. The proof of Lemma 2.1 shows that both $\mathcal{M}$ and $\mathcal{L}$ are sectorial in $\mathfrak{C}$. Let $\mu_0 \in \mathbb{R}$ be such that if $\mu \in \mathbb{C}$ and Real $\mu \geq \mu_0$, then given $f \in \mathfrak{C}$, $(\mathcal{L} - \mu I)g = f$ and $(\mathcal{M} - \mu I)h = f$ are solvable for g and h respectively. Then, keeping in mind that functions in $\widetilde{\mathfrak{C}}$ are vector-valued, an argument similar to that in the proof of [11, p 92, Proposition 3.1.18] yields the existence of $K > 0$, independent of $\mu \in \mathbb{C}$ with Real $\mu \geq \mu_0$, such that

$$\|\mu(\mu I - \mathcal{L})^{-1}\|_{L(\widetilde{\mathfrak{C}}^1, \widetilde{\mathfrak{C}}^1)} < K \quad \text{if} \quad \text{Real } \mu \geq \mu_0.$$

The result follows from [11, p 43, Proposition 2.1.11]. $\square$

We turn now to the spectral analysis of $\mathcal{L}$. Denote the spectrum of $\mathcal{L}$ by $\sigma(\mathcal{L})$ and the essential spectrum by $\sigma_{ess}(\mathcal{L})$. (Here, as in [8], the essential spectrum of $\mathcal{L}$ is the complement, in $\sigma(\mathcal{L})$, of the set of those eigenvalues of finite (algebraic) multiplicity[1] which are isolated points of $\sigma(\mathcal{L})$.) Of crucial importance is the following lemma concerning the eigenvalues of the "asymptotic form of $\mathcal{L}$ at infinity". It makes critical use of the bistability condition **(f3)**. We define

$$C^\pm = \lim_{x \to \pm\infty} C(x) = cI + d_p f[E^\pm, 0] \quad \text{and} \quad B^\pm = \lim_{x \to \pm\infty} B(x) = d_q f[E^\pm, 0]. \tag{9}$$

Lemma 2.3 *Suppose that there exist $\tau \in \mathbb{R}, \lambda \in \mathbb{C}$ and $z \in \mathbb{C}^N$ such that*

$$(-\tau^2 A + i\tau C^+ + B^+)z = \lambda z. \tag{10}$$

Then Real $\lambda < 0$. The same conclusion holds if C^+, B^+ are replaced by C^-, B^- in (10).

Proof. By condition **(f3)**, all the eigenvalues of $B^\pm$ lie in the open left-half complex plane. By condition **(f1)**, $C^\pm$ are diagonal and by condition **(f2)**, $B^\pm$ each have positive off-diagonal elements. So the result follows immediately from [14, p 234, Lemma 4.1]. $\square$

Lemma 2.4 *$\sigma_{ess}(\mathcal{L}) \neq \emptyset$, and there exists $\beta > 0$ such that if $\lambda \in \sigma_{ess}(\mathcal{L})$ then Real $\lambda < -\beta$.*

[1] An eigenvalue λ_0 which is an isolated point of the spectrum is said to have finite (algebraic) multiplicity if $\mathcal{P}\mathfrak{C}$ is finite-dimensional, where $\mathcal{P}$ is the linear operator defined by $\mathcal{P} = \frac{1}{2\pi i} \int_{\partial\Omega} (\xi I - \mathcal{L})^{-1} \, d\xi$, Ω being a ball in $\mathbb{C}$, centre λ_0, such that $\sigma(\mathcal{L}) \cap \bar{\Omega} = \{\lambda_0\}$ [9, p 181].

Proof. Let

$$S^{\pm} = \{\lambda \in \mathbb{C} : \det\,(-\tau^2 A + i\tau C^{\pm} + B^{\pm} - \lambda I) = 0 \ \text{ for some } \ \tau \in \mathbb{R}\}. \tag{11}$$

Then Lemma 2.3 shows that

$$\lambda \in S^+ \cup S^- \Rightarrow \ \text{Real}\ \lambda < 0.$$

[8, p 140, Theorem A.2] yields that $S^{\pm}$ each consists of a finite number of algebraic curves parametrised by a real number σ, which are asymptotically parabolic : $\lambda(\sigma) = -\sigma^2\alpha + O(\sigma)$ as $\sigma \to \infty$, where α is on the diagonal of A. Moreover, $S^+ \cup S^- \subset \sigma_{ess}(\mathcal{L})$ and $\sigma_{ess}(\mathcal{L}) \subset \Lambda$, where $\mathbb{C}\backslash\Lambda$ is the component of $\mathbb{C}\backslash(S^+ \cup S^-)$ which contains the right-half plane.

Since $S^{\pm}$ are contained in the open left-half plane, Λ is also. Moreover, $S^{\pm}$ each consist of a finite number of algebraic curves parametrised by σ, the real parts of which tend to $-\infty$ as $\sigma \to \pm\infty$. Whence Λ is bounded away from the imaginary axis. The result follows. $\square$

We next show, using Lemma 2.3, that the bistability condition **(f3)** implies that bounded solutions of certain equations must decay at infinity.

Lemma 2.5 *Suppose that there exist $\lambda \in \mathbb{C}$, Real $\lambda \geq 0$ and $g \in \widetilde{\mathfrak{C}}^2$ such that $\mathcal{L}g = \lambda g + \psi_0$, where $\psi_0 \in \mathfrak{C}$ is such that $\psi_0(x) \to 0$ as $|x| \to \infty$. Then $\|g(x)\| \to 0$ as $|x| \to \infty$. If $\psi_0 \equiv 0$, then there exist $M, \omega > 0$ such that $\|g(x)\| \leq Me^{-\omega|x|}$ for all $x \in \mathbb{R}$.*

Proof. Define $\hat{h} = \begin{pmatrix} g \\ g' \end{pmatrix}$, $M^+ = \begin{pmatrix} 0 & I \\ -A^{-1}\{B^+ - \lambda I\} & -A^{-1}C^+ \end{pmatrix}$, $\hat{r}(x) = \hat{H}(x)\hat{h}(x) + \begin{pmatrix} 0 \\ \psi_0(x) \end{pmatrix}$, where $\hat{H}(x) = \begin{pmatrix} 0 & 0 \\ -A^{-1}\{B(x) - B^+\} & -A^{-1}\{C(x) - C^+\} \end{pmatrix}$. Then $\hat{h}'(x) = M^+\hat{h}(x) + \hat{r}(x)$, $x \in \mathbb{R}$, where $\hat{h}$ is bounded on $\mathbb{R}$, and $\hat{r}(x) \to 0$ as $x \to \infty$. By Lemma 2.3, M^+ has no purely imaginary eigenvalues. So, as in the proof of [4, p 330, Theorem 4.1], there exist $K, \alpha, \sigma > 0$, a real nonsingular matrix $P \in M^{2N\times 2N}$ and operators $U_1(t), U_2(t)$ such that

$$\|U_1(x)\| \leq Ke^{-\alpha x}, \quad x \geq 0 \ \text{ and } \ \|U_2(x)\| \leq Ke^{\sigma x}, \quad x \leq 0, \tag{12}$$

and $h = P\hat{h}$, $r = P\hat{r}$ satisfy

$$h(x) = U_1(x)h(0) + U_2(x)k + \int_0^x U_1(x-s)r(s)\ ds - \int_x^\infty U_2(x-s)r(s)\ ds$$

where

$$k = h(0) + \int_0^\infty U_2(-s)r(s)\ ds.$$

Estimates (12) together with the facts that $\hat{h}$ is bounded and $\hat{r} \to 0$ as $x \to \infty$ yield that $h(x) \to 0$ as $x \to \infty$. The exponential decay in the case $\psi_0 \equiv 0$ follows from the proof of [4, p 330, Theorem 4.1]. $\square$

Remark. Clearly, due to translation invariance, $\mathcal{L}w' = 0$, where w' is the derivative of the travelling wave w. By hypothesis **(TW)**, w' is bounded on $\mathbb{R}$, so Lemma 2.5 yields that w' decays exponentially to zero at $\pm\infty$. Also by **(TW)**, $w'(x) > 0$ for all $x \in \mathbb{R}$. Thus $\mathcal{L}u = 0$ has a positive solution which decays exponentially to zero at infinity. Further, Lemma 2.4 shows that zero is not in the essential spectrum of $\mathcal{L}$, so it must be an isolated point of the spectrum and an eigenvalue of finite multiplicity.

Lemma 2.6 *(i) For $\lambda \in \mathbb{C}\backslash\{0\}$ with Real $\lambda \geq 0$, there are no non-zero solutions of the equation*

$$\mathcal{L}g = \lambda g, \quad g \in \widetilde{\mathfrak{C}}^2. \tag{13}$$

(ii) Let $g \in \widetilde{\mathfrak{C}}^2$ be a solution of $\mathcal{L}g = 0$. Then there exists $k \in \mathbb{R}$ such that $g = kw'$.

Proof. We aim to apply [14, p 208, Theorem 5.1]. For this, note that **(f2)** and **(f3)** imply that the matrix is irreducible in the functional sense (defined in [14, p 208]); this follows from **(f2)** alone when $N \geq 2$. Now let $\lambda \in \mathbb{C}$, Real $\lambda \geq 0$, and suppose that $g \in \widetilde{\mathfrak{C}}^2$ satisfies $\mathcal{L}g = \lambda g$. That $\|g(x)\| \to 0$ as $|x| \to \infty$ follows from Lemma 2.5 with $\psi_0 \equiv 0$. The remark preceding this theorem together with [14, p 208, parts (1) and (2) of Theorem 5.1] then yield **(i)** and **(ii)**. □

Proposition 2.7 *There exists $\gamma > 0$ such that if $\lambda \in \mathbb{C}$ belongs to $\sigma(\mathcal{L})\backslash\{0\}$, then Real $\lambda < -\gamma$.*

Proof. Lemma 2.4 and Lemma 2.6 **(i)** show that any non-zero point of $\sigma(\mathcal{L})$ lies in the open left-half complex plane. If there is a sequence $\{\lambda_n\} \subset \sigma(\mathcal{L})\backslash\{0\}$ such that Real $\lambda_n \uparrow 0$ as $n \to \infty$, then by Lemma 2.1, {Imag λ_n} is bounded. Whence there is a subsequence $\{\lambda_k\}$ and $\mu \in \sigma(\mathcal{L})$ (a closed set), Real $\mu = 0$, such that $\lambda_k \to \mu$ as $k \to \infty$. But this contradicts Lemma 2.4. □

Lemma 2.6 **(ii)** shows that the nullspace of $\mathcal{L}$ is one-dimensional. We need additional information to exploit this. Recall that zero is an isolated eigenvalue of $\mathcal{L}$. Let Ω denote a ball in $\mathbb{C}$ with centre zero such that $\sigma(\mathcal{L}) \cap \overline{\Omega} = \{0\}$. Then for $\lambda \in \partial\Omega$, $(\lambda I - \mathcal{L})^{-1} : \mathfrak{C} \to \mathfrak{C}$ is a bounded linear operator; a bounded linear operator $\mathcal{P}$ is defined by

$$\mathcal{P} = \frac{1}{2\pi i} \int_{\partial\Omega} (\xi I - \mathcal{L})^{-1} \, d\xi, \tag{14}$$

(see [9, p 178] or [11, p 402]). Let $X = \mathfrak{C}, X_1 = \mathcal{P}X$ and $X_2 = (I - \mathcal{P})X$. [8, p 30, Theorem 1.5.2] and [11, p 402, Proposition A.1.2] show that $\mathcal{P}$ is a projection, $X = X_1 \oplus X_2$ and $\mathcal{P}X$ is a subset of the domain of $\mathcal{L}^n$ for each n. Moreover, if $\mathcal{L}_j$ is the restriction of $\mathcal{L}$ to $X_j \cap \mathfrak{C}^2$, then

$$\begin{aligned} \mathcal{L}_1 &: X_1 \to X_1 \quad \text{is bounded,} \quad \sigma(\mathcal{L}_1) = \{0\} \;\text{ and} \\ \mathcal{L}_2 &: X_2 \cap \mathfrak{C}^2 \subset X_2 \to X_2, \;\; \sigma(\mathcal{L}_2) = \sigma(\mathcal{L})\backslash\{0\} \;\; (\neq \emptyset, \text{ by Lemma 2.4}). \end{aligned}$$

Note that since $\mathcal{P}, I - \mathcal{P}$ are bounded operators by definition, X_1 and X_2 are closed subspaces of X.

Lemma 2.8 *$X_1 = span\{w'\}$ and there exists $w^* \in X^*$ such that*

$$\mathcal{P}g = w^*(g)w' \;\; \textit{for each} \;\; g \in X, \quad \textit{and} \;\; w^*(w') = 1. \tag{15}$$

Proof. [11, p 405, Proposition A.2.2] shows that ker $\mathcal{L} \subset X_1$. Since $0 \notin \sigma_{ess}(\mathcal{L})$, X_1 is finite-dimensional (see the footnote following the definition of $\sigma_{ess}(\mathcal{L})$). So $\sigma(\mathcal{L}_1)$ consists entirely of eigenvalues, the number of which, counted according to algebraic multiplicity, equals the dimension of X_1. It is shown in [14, p 210, proof of Theorem 5.1 (3)] that Range $\mathcal{L} \cap$ span $\{w'\} = 0$. Thus zero is an eigenvalue of $\mathcal{L}_1$ of multiplicity one, whence ker $\mathcal{L} = X_1$. Since $\mathcal{P}$ is a bounded projection, the existence of w^* as in the statement of the lemma follows. □

We will need two estimates on the behaviour of $\mathcal{L}_2$. Define $\gamma_0 = -\sup\ \{$ Real $z : z \in \sigma(\mathcal{L}_2)\}$. By Proposition 2.7, $\gamma_0 > 0$.

Lemma 2.9 *Given $\epsilon \in (0, \gamma_0)$, there exists $M_\epsilon \geq 1$ such that for $g \in X_2 \cap \mathfrak{C}^1, t > 0$,*

$$\|e^{t\mathcal{L}_2} g\|_{\mathfrak{C}^1} \leq M_\epsilon t^{-\frac{1}{2}} e^{-\gamma_\epsilon t} \|g\|_{\mathfrak{C}} \tag{16}$$

and

$$\|e^{t\mathcal{L}_2} g\|_{\mathfrak{C}^1} \leq M_\epsilon e^{-\gamma_\epsilon t} \|g\|_{\mathfrak{C}^1}, \tag{17}$$

where $\gamma_\epsilon = \gamma_0 - \epsilon$.

Proof. Lemma 2.2 implies that the part of $\mathcal{L}$ in $\mathfrak{C}^1$ generates an analytic semigroup in the Banach space $\mathfrak{C}^1$. So there exist $M > 0$ and $\omega \in \mathbb{R}$ such that for each $t > 0, g \in \mathfrak{C}^1$,

$$\|e^{t\mathcal{L}} g\|_{\mathfrak{C}^1} \leq M e^{\omega t} \|g\|_{\mathfrak{C}^1}. \tag{18}$$

Fix $\epsilon \in (0, \gamma_0)$. We appeal to [11], in the notation of which, let $\alpha = \frac{1}{2}$ and $n = 0$. The spaces $D_{\mathcal{L}}(\frac{1}{2}, p), 1 \leq p \leq \infty$ are defined in [11, p 45]; note the last remark on that page. Now observe that

$$D_{\mathcal{L}}(\tfrac{1}{2}, 1) \hookrightarrow \mathfrak{C}^1. \tag{19}$$

This follows from Landau's inequality, [11, p 46, Proposition 2.2.2 and p 24, Theorem 1.2.13 with $\theta = \frac{1}{2}$]. This and [11, p 59, Proposition 2.3.3 with $\beta = \frac{1}{2}$ and $p = 1$] together yield the existence of $\hat{M} > 0$ such that for each $g \in X_2 \cap \mathfrak{C}^1$,

$$\|e^{t\mathcal{L}_2} g\|_{\mathfrak{C}^1} \leq \hat{M} t^{-\frac{1}{2}} e^{-\gamma_\epsilon t} \|g\|_{\mathfrak{C}} \quad \text{for each } t > 0. \tag{20}$$

In addition,

$$\mathfrak{C}^1 \hookrightarrow D_{\mathcal{L}}(\tfrac{1}{2}, \infty) \quad \text{and} \quad D_{\mathcal{L}}(\beta, \infty) \hookrightarrow \mathfrak{C}^1, \beta \in (\tfrac{1}{2}, 1), \tag{21}$$

by [11, p 86, Theorem 3.1.12 with $\theta = \frac{1}{2}$ and $\theta = \beta$ respectively]. [11, p 59, Proposition 2.3.3 with $\beta \in (\frac{1}{2}, 1), p = \infty$] and (21) give the existence of $\hat{M} > 0$ such that for each $g \in X_2 \cap \mathfrak{C}^1$,

$$\begin{aligned} \|e^{t\mathcal{L}_2} g\|_{\mathfrak{C}^1} &\leq \hat{M} t^{\frac{1}{2}-\beta} e^{-\gamma_\epsilon t} \|g\|_{\mathfrak{C}^1}, \quad \text{for each } \ t > 0, \\ &\leq \hat{M} e^{-\gamma_\epsilon t} \|g\|_{\mathfrak{C}^1} \quad \text{when } \ t \geq 1. \end{aligned} \tag{22}$$

It follows from (18) and (22) that there exists $\tilde{M} > 0$ such that

$$\|e^{t\mathcal{L}_2} g\|_{\mathfrak{C}^1} \leq \tilde{M} e^{-\gamma_\epsilon t} \|g\|_{\mathfrak{C}^1} \quad \text{for all } \ t > 0. \tag{23}$$

(16) and (17) follow from (20) and (23). □

3 Local stability

It is useful to formulate (6) as an abstract ordinary differential equation. Let $T > 0$ and let $v \in C(\mathbb{R} \times [0, T], \mathbb{R}^N)$ be such that v, v_t, v_x and v_{xx} are bounded and uniformly continuous on $\mathbb{R} \times (0, T)$. Define $y(t)(x) = v(x, t) - w(x), \quad (x, t) \in \mathbb{R} \times [0, T]$, where w is the travelling wave introduced in **(TW)**. Then v satisfies (6) if and only if $y \in C^1((0, T), \mathfrak{C}) \cap C((0, T), \mathfrak{C}^2)$ satisfies

$$y'(t) = \mathcal{L}(y(t)) + \mathcal{R}(y(t)), \quad t \in (0, T) \tag{24}$$

where $\mathcal{R} : \mathfrak{C}^1 \to \mathfrak{C}$ is given by

$$\mathcal{R}(y) = f(w + y, w' + y') - f(w, w') - d_p f[w, w']y' - d_q f[w, w']y, \quad y \in \mathfrak{C}^1.$$

Note that $\mathcal{R}$ is continuously differentiable, and that $\|\mathcal{R}(y)\|_{\mathfrak{C}}/\|y\|_{\mathfrak{C}^1} \to 0$ as $\|y\|_{\mathfrak{C}^1} \to 0$.

Following [8, p 108], we adopt an elementary approach to proving local stability, based on the variation of constants formula and the estimates of Lemma 2.9. An alternative is to use centre-manifold theory and the existence of foliations - see [1], [2], [3].

Theorem 3.1 *Let $\epsilon \in (0, \gamma_0)$. Then there exist $\nu_\epsilon > 0, K_\epsilon > 0$ and $\delta_\epsilon > 0$ such that if $\phi \in \mathfrak{C}^1$ satisfies*

$$\|\phi - w(\cdot + \chi_0)\|_{\mathfrak{C}^1} < \nu_\epsilon \tag{25}$$

for some $\chi_0 \in \mathbb{R}$, then there exists $\chi_\infty \in [\chi_0 - \delta_\epsilon, \chi_0 + \delta_\epsilon]$ such that

$$\|v^\phi(\cdot, t) - w(\cdot + \chi_\infty)\|_{\mathfrak{C}^1} \leq K_\epsilon e^{-\gamma_\epsilon t}, \quad t > 0. \tag{26}$$

Note that K_ϵ and $\delta_\epsilon > 0$ are independent of the exact choice of ϕ, χ_0 satisfying (25).

Proof. We first prove a convergence result for (24), and then deduce Theorem 3.1 by interpreting this in terms of (6) and the travelling wave w. The idea for the proof comes from [8, p 108, Exercise 6]. For $\chi \in \mathbb{R}$, define $\hat{w} : \mathbb{R} \to \mathfrak{C}^1$ by $\hat{w}(\chi)(x) = w(x + \chi) - w(x)$, $x \in \mathbb{R}$. Then $\hat{w}(0) = 0$, and for each $\chi \in \mathbb{R}$, $\mathcal{L}\hat{w}(\chi) + \mathcal{R}(\hat{w}(\chi)) = 0$, since $w(\cdot + \chi)$ is a stationary solution of (6). Since w satisfies **(TW)** and $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$, $w \in \mathfrak{C}^3$. So $\hat{w} : \mathbb{R} \to \mathfrak{C}^1$ is twice continuously differentiable, and

$$d\hat{w}[\chi_0]\chi = \chi w'(\cdot + \chi_0) \text{ for each } \chi_0, \chi \in \mathbb{R}. \tag{27}$$

Let $\mathcal{H}(y, \chi) = w^*(y - \hat{w}(\chi)) \in \mathbb{R}$, $(y, \chi) \in \mathfrak{C}^1 \times \mathbb{R}$, where w^* is as in Lemma 2.8. Then $\mathcal{H}$ is continuously differentiable, $\mathcal{H}(0, 0) = 0$ and $d_\chi\mathcal{H}[0, 0]\chi = -\chi$ for each $\chi \in \mathbb{R}$. So it follows from the implicit function theorem that there is an open ball $B_{\mathfrak{C}^1}(\rho_0)$ in $\mathfrak{C}^1$ (centre 0, radius ρ_0), an open neighbourhood $(-\delta_0, \delta_0)$ of 0 in $\mathbb{R}$ and a continuously differentiable function $\zeta : B_{\mathfrak{C}^1}(\rho_0) \to (-\delta_0, \delta_0)$ such that $\zeta(0) = 0$, $\mathcal{H}(y, \zeta(y)) = 0$ for $y \in B_{\mathfrak{C}^1}(\rho_0)$, and if $\mathcal{H}(y, \chi) = 0$ for some $y \in B_{\mathfrak{C}^1}(\rho_0), \chi \in (-\delta_0, \delta_0)$, then $\chi = \zeta(y)$. By (15), we can choose $\rho_0 > 0$ smaller if necessary so that $w^*(w'(\cdot+\chi)) > \frac{1}{2}$ whenever $\chi = \zeta(y)$ for some $y \in B_{\mathfrak{C}^1}(\rho_0)$.

Proposition A.3 (Appendix) ensures that given initial data $y_0 \in \mathfrak{C}^1$, there is a unique local classical solution $y : (0, \tau(y_0)) \to \mathfrak{C}^2$ of (24) such that $\|y(t) - y_0\|_{\mathfrak{C}^1} \to 0$ as $t \to 0$. For $y_0 \in B_{\mathfrak{C}^1}(\rho_0)$, let $\hat{t} \in (0, \tau(y_0))$ be such that $y(t) \in B_{\mathfrak{C}^1}(\rho_0)$ for each $t \in [0, \hat{t}]$. For such t, define $\chi(t) = \zeta(y(t))$, where ζ is as given by the implicit function theorem above. Then $\chi(t) \in (-\delta_0, \delta_0)$ and $w^*(y(t)) = w^*(\hat{w}(\chi(t)))$. Define $\hat{y}(t) = y(t) - \hat{w}(\chi(t))$. Since $w^*(\hat{y}(t)) = 0$, $\hat{y}(t) \in X_2$ (where X_2 is as defined before Lemma 2.8). Note that $\hat{w}(\chi(\cdot)) = \hat{w}(\zeta(y(\cdot)))$ and $\hat{y}(\cdot)$ are both continuously differentiable on $(0, \hat{t})$, and since $y \in C^1((0, \hat{t}), \mathfrak{C})$ and X_2 is a closed subspace of $\mathfrak{C}$, $\hat{y}'(t) \in X_2$ for $0 < t < \hat{t}$.

Acting on (24) with w^* and using (27), the fact that $\hat{w}(\chi)$ is a stationary solution of (24) for each χ and the properties of w^* together yield that for $0 < t < \hat{t}$,

$$\chi'(t)w^*(w'(\cdot + \chi(t))) = w^*(\mathcal{R}(\hat{w}(\chi(t)) + \hat{y}(t)) - \mathcal{R}(\hat{w}(\chi(t)))). \tag{28}$$

So

$$\chi'(t) = \Phi(\chi(t), \hat{y}(t)), \; t \in (0, \hat{t}), \tag{29}$$

where we define

$$\Phi(\chi, \hat{y}) = \frac{w^*(\mathcal{R}(\hat{w}(\chi) + \hat{y}) - \mathcal{R}(\hat{w}(\chi)))}{w^*(w'(\cdot + \chi))}, \; (\chi, \hat{y}) \in \mathbb{R} \times \mathfrak{C}^1. \tag{30}$$

Similarly, acting on (24) with $I-\mathcal{P}$ (see (14)) gives that

$$\hat{y}'(t) = \mathcal{L}_2\hat{y}(t) + \Psi(\chi(t),\hat{y}(t)), \quad t \in (0,\hat{t}), \tag{31}$$

where

$$\Psi(\chi,\hat{y}) = (I-\mathcal{P})\{\mathcal{R}(\hat{w}(\chi)+\hat{y}) - \mathcal{R}(\hat{w}(\chi))\} - (I-\mathcal{P})d\hat{w}[\chi]\Phi(\chi,\hat{y}). \tag{32}$$

Now for $\hat{y} \in \mathfrak{C}^1$ and $\chi \in \mathbb{R}$ with $|\chi| \le 1$ and small enough that $w^*(w'(\cdot+\chi)) > \frac{1}{2}$,

$$|\Phi(\chi,\hat{y})| \le 2\|w^*\|_{\mathfrak{C}^*}K(\chi,\hat{y})\|\hat{y}\|_{\mathfrak{C}^1}, \quad \text{where} \quad K(\chi,\hat{y}) = \sup_{0\le\theta\le 1}\{\|d\mathcal{R}[\hat{w}(\chi)+\theta\hat{y}]\|_{L(\mathfrak{C}^1,\mathfrak{C})}\}.$$

Since $d\mathcal{R}[0]=0$, $K(\chi,\hat{y}) \to 0$ as $|\chi| + \|\hat{y}\|_{\mathfrak{C}^1} \to 0$. Also,

$$\|\Psi(\chi,\hat{y})\|_{\mathfrak{C}} \le \|I-\mathcal{P}\|_{L(\mathfrak{C},\mathfrak{C})}K(\chi,\hat{y})\|\hat{y}\|_{\mathfrak{C}^1} + \|I-\mathcal{P}\|_{L(\mathfrak{C},\mathfrak{C})}\|d\hat{w}[\chi]\|_{L(\mathbb{R},\mathfrak{C}^1)}|\Phi(\chi,\hat{y})|. \tag{33}$$

So, since $\|d\hat{w}[\chi]\|_{L(\mathbb{R},\mathfrak{C}^1)}$ is bounded independently of $|\chi| \le 1$, there exists a constant $\hat{K} > 0$ such that

$$|\Phi(\chi,\hat{y})| + \|\Psi(\chi,\hat{y})\|_{\mathfrak{C}} \le \hat{K}K(\chi,\hat{y})\|\hat{y}\|_{\mathfrak{C}^1}, \quad \text{where } K(\chi,\hat{y}) \to 0 \text{ as } |\chi| + \|\hat{y}\|_{\mathfrak{C}^1} \to 0. \tag{34}$$

Henceforth fix $\epsilon \in (0,\gamma_0)$. Choose $\sigma_\epsilon > 0$ so that

$$M_{\frac{\epsilon}{2}}\sigma_\epsilon \int_0^\infty s^{-\frac{1}{2}}e^{-(\gamma_{\frac{\epsilon}{2}}-\gamma_\epsilon)s}\,ds = M_{\frac{\epsilon}{2}}\sigma_\epsilon \int_0^\infty s^{-\frac{1}{2}}e^{-\frac{\epsilon}{2}s}\,ds < \frac{1}{2}, \tag{35}$$

where $M_{\frac{\epsilon}{2}} \ge 1$ is as in Lemma 2.9. Let $\tilde{K} > 0$ be such that $K(\chi,\hat{y}) < \tilde{K}$ whenever $|\chi| < \delta_0$ and $\|\hat{y}\|_{\mathfrak{C}^1} < \rho_0$. Now using (34), we can choose $\rho_\epsilon \in (0,\rho_0)$, $\delta_\epsilon \in (0,\delta_0)$ such that $\rho_\epsilon < \frac{\gamma_\epsilon^2}{2\hat{K}\tilde{K}}$ and

$$\|\Psi(\chi,\hat{y})\|_{\mathfrak{C}} \le \sigma_\epsilon\|\hat{y}\|_{\mathfrak{C}^1}, \|\hat{w}(\chi)+\hat{y}\|_{\mathfrak{C}^1} \le \frac{\rho_0}{2} \text{ for all } (\chi,\hat{y}) \text{ with } |\chi| \le \delta_\epsilon \text{ and } \|\hat{y}\|_{\mathfrak{C}^1} \le \rho_\epsilon. \tag{36}$$

Let $\nu_\epsilon \in (0,\rho_0)$ be such that

$$\|y_0\|_{\mathfrak{C}^1} < \nu_\epsilon \Rightarrow |\zeta(y_0)| < \delta_\epsilon/2 \text{ and } \|y_0\|_{\mathfrak{C}^1} + \|\hat{w}(\zeta(y_0))\|_{\mathfrak{C}^1} < \rho_\epsilon/(2M_{\frac{\epsilon}{2}}). \tag{37}$$

Fix initial data $y_0 \in \mathfrak{C}^1$ with $\|y_0\|_{\mathfrak{C}^1} < \nu_\epsilon$. Define $t_0 = \sup_{0\le t<\tau(y_0)}\{t : y(s) \in B_{\mathfrak{C}^1}(\rho_0) \text{ for all } s \in [0,t]\}$. For $t \in [0,t_0)$, $\chi(t) = \zeta(y(t))$ and $\hat{y}(t) = y(t) - \hat{w}(\chi(t))$ are well-defined and have the properties described above. By the choice of ν_ϵ, $|\chi(0)| < \frac{\delta_\epsilon}{2}$ and $\|\hat{y}(0)\|_{\mathfrak{C}^1} < \frac{\rho_\epsilon}{2M_{\frac{\epsilon}{2}}}$. Define $m(t) = \sup_{0\le s\le t}\{e^{\gamma_\epsilon s}\|\hat{y}(s)\|_{\mathfrak{C}^1}\}$, $t \in [0,t_0)$. Then since $\hat{y}$ satisfies (31) and $\gamma_\epsilon = \gamma_0 - \epsilon$, it follows from the variation of constants formula, Lemma 2.9, (35) and (36) that for $0 \le s \le t < t_0$,

$$\begin{aligned}
e^{\gamma_\epsilon s}\|\hat{y}(s)\|_{\mathfrak{C}^1} &= e^{\gamma_\epsilon s}\left\|e^{s\mathcal{L}_2}\hat{y}(0) + \int_0^s e^{(s-\tilde{s})\mathcal{L}_2}\Psi(\chi(\tilde{s}),\hat{y}(\tilde{s}))\,d\tilde{s}\right\|_{\mathfrak{C}^1} \\
&\le M_{\frac{\epsilon}{2}}e^{(\gamma_\epsilon-\gamma_{\frac{\epsilon}{2}})s}\|\hat{y}(0)\|_{\mathfrak{C}^1} + e^{\gamma_\epsilon s}\sigma_\epsilon M_{\frac{\epsilon}{2}}\int_0^s (s-\tilde{s})^{-\frac{1}{2}}e^{-\gamma_{\frac{\epsilon}{2}}(s-\tilde{s})}\|\hat{y}(\tilde{s})\|_{\mathfrak{C}^1}\,d\tilde{s} \\
&\le M_{\frac{\epsilon}{2}}\|\hat{y}(0)\|_{\mathfrak{C}^1} + \tfrac{1}{2}m(t).
\end{aligned}$$

Whence $m(t) \le 2M_{\frac{\epsilon}{2}}\|\hat{y}(0)\|_{\mathfrak{C}^1}$ for each $t \in [0,t_0)$. It follows, using (29), (34), that

$$\|\hat{y}(t)\|_{\mathfrak{C}^1} \le \rho_\epsilon e^{-\gamma_\epsilon t} \quad \text{and} \quad |\chi'(t)| = |\Phi(\chi(t),\hat{y}(t))| \le \hat{K}\tilde{K}\rho_\epsilon e^{-\gamma_\epsilon t}, \quad t \in (0,t_0). \tag{38}$$

This, together with the facts that $|\chi(0)| < \delta_\epsilon/2$ and $\rho_\epsilon < \frac{\gamma_\epsilon^2}{2\hat{K}\tilde{K}}$, yields that for each $t \in [0, t_0)$,

$$|\chi(t)| \leq \delta_\epsilon/2 + \hat{K}\tilde{K}\rho_\epsilon\gamma_\epsilon^{-1}[1 - e^{-\gamma_\epsilon t}] \ < \ \delta_\epsilon. \tag{39}$$

Now it follows from the definition of t_0, (36), (38) and (39) that $t_0 = \tau(y_0)$. And Proposition A.4 (Appendix) shows that if $\tau(y_0) < \infty$, then $\sup_{0\leq s\leq t} \|y(s)\|_{\mathfrak{C}} \to \infty$ as $t \uparrow \tau(y_0)$. So $t_0 = \tau(y_0) = \infty$, and (38) and (39) hold for all $t \geq 0$. Since $|\chi'(\cdot)| \in L^1((0,\infty), \mathbb{R})$ and $|\chi(t)| \leq \delta_\epsilon$ for all $t \geq 0$, there exists $\hat{\chi} \in [-\delta_\epsilon, \delta_\epsilon]$ such that

$$|\hat{\chi} - \chi(t)| \leq \hat{K}\tilde{K}\rho_\epsilon{\gamma_\epsilon}^{-1}e^{-\gamma_\epsilon t}, \quad t > 0. \tag{40}$$

We now rewrite (38) and (40) in terms of the travelling wave w. Recall that y is a solution of (24) with initial data y_0 if and only if $v^\phi(\cdot, t) = y(t) + w$ is a solution of (6) with initial data $\phi = y_0 + w$, and that $\hat{w}(\chi)(x) = w(x + \chi) - w(x)$ for $x, \chi \in \mathbb{R}$. So $\|y_0\|_{\mathfrak{C}^1} = \|\phi - w\|_{\mathfrak{C}^1}$, and

$$\|\hat{y}(t)\|_{\mathfrak{C}^1} = \|y(t) - \hat{w}(\chi(t))\|_{\mathfrak{C}^1} = \|v^\phi(\cdot, t) - w(\cdot + \chi(t))\|_{\mathfrak{C}^1}. \tag{41}$$

Hence if $\|\phi - w\|_{\mathfrak{C}^1} \leq \nu_\epsilon$, (38) and (40) give that

$$\begin{aligned} \|v^\phi(\cdot, t) - w(\cdot + \hat{\chi})\|_{\mathfrak{C}^1} &\leq \|v^\phi(\cdot, t) - w(\cdot + \chi(t))\|_{\mathfrak{C}^1} + \|w(\cdot + \chi(t)) - w(\cdot + \hat{\chi})\|_{\mathfrak{C}^1} \\ &\leq \rho_\epsilon e^{-\gamma_\epsilon t} + |\chi(t) - \hat{\chi}|\|w'\|_{\mathfrak{C}} \ \leq \ K_\epsilon e^{-\gamma_\epsilon t}, \end{aligned}$$

where $K_\epsilon = \rho_\epsilon\{1 + \hat{K}\tilde{K}\gamma_\epsilon^{-1}\|w'\|_{\mathfrak{C}}\}$. To complete the proof, note that if $\phi \in \mathfrak{C}^1$ satisfies $\|\phi - w(\cdot + \chi_0)\|_{\mathfrak{C}^1} < \nu_\epsilon$ for some $\chi_0 \in \mathbb{R}$, then $\|\phi(\cdot - \chi_0) - w(\cdot)\|_{\mathfrak{C}^1} < \nu_\epsilon$. The above analysis immediately implies that $\|v^\phi(\cdot, t) - w(\cdot + \hat{\chi} + \chi_0)\|_{\mathfrak{C}^1} \leq K_\epsilon e^{-\gamma_\epsilon t}$ for all $t \geq 0$. The result follows. □

4 Global stability for monotone initial data

We turn now to the global stability of the wave w. Note first that if $\phi \in \mathfrak{C}^1$ satisfies $E^- \leq \phi(x) \leq E^+$ for all $x \in \mathbb{R}$, then it follows from Theorem A.7 (Appendix) that the initial value problem (6) has a unique classical solution v^ϕ that exists for all time, and that $E^- \leq v^\phi(x,t) \leq E^+$ for all $x \in \mathbb{R}$, $t \geq 0$.

In this section, we consider the initial-value problem (6) with initial data $\phi \in \mathfrak{C}^1$ satisfying the following conditions :

(ϕ1) $\phi(x) \to E^\pm$ as $x \to \pm\infty$, and $\phi'(x) \to 0$ as $|x| \to \infty$,

(ϕ2) $\phi'(x) \geq 0$ for each $x \in \mathbb{R}$.

Our approach is similar to that of [14, pp 245-248, Theorem 6.1].

Theorem 4.1 *Let $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ satisfy* **(f1) - (f4)** *and $\phi \in \mathfrak{C}^1$ satisfy* (ϕ1) − (ϕ2)*. Then there exists $\chi_\infty \in \mathbb{R}$ such that for each $\epsilon \in (0, \gamma_0)$, there exists $N_\epsilon > 0$ such that the solution v^ϕ of (6) with initial data ϕ satisfies*

$$\|v^\phi(\cdot, t) - w(\cdot + \chi_\infty)\|_{\mathfrak{C}^1} \leq N_\epsilon e^{-\gamma_\epsilon t}, \quad \textit{for all} \ \ t > 0. \tag{42}$$

Proof. The idea is to construct a function ϕ^*, from ϕ and the wave w, such that the solution v^{ϕ^*} of (6) satisfies (42), and then to use a homotopy argument to deduce the corresponding result for ϕ.

Fix $\epsilon \in (0, \gamma_0)$. We begin with the construction of ϕ^*. Let ν_ϵ be as in (37). Choose $\eta_1 > 0$ sufficiently large that

$$\pm x \geq +\eta_1 \Rightarrow \|\phi(x) - E^{\pm}\|,\ \|w(x) - E^{\pm}\|,\ \|w'(x)\|, \|\phi'(x)\| < \frac{\nu_\epsilon}{4}. \tag{43}$$

Choose $\eta_2 > \eta_1 + 1$ so that $\phi(\eta_2) > w(\eta_1)$ and $\phi(-\eta_2) < w(-\eta_1)$. Define $\phi^* : \mathbb{R} \to \mathbb{R}^N$ by $\phi^*(x) = w(x)$ for $|x| \leq \eta_1$ and $\phi^*(x) = \phi(x)$ for $|x| \geq \eta_2$; for $|x| \in [\eta_1, \eta_2]$, define $\phi^*(x)$ so that $\phi^* \in \mathfrak{C}^1$ is increasing and $\|(\phi^*)'(x)\| < \nu_\epsilon/4$ for each $x, |x| \geq \eta_1$. By construction,

$$\|\phi^* - w\|_{\mathfrak{C}^1} < \frac{\nu_\epsilon}{2}. \tag{44}$$

Here is the construction that underlies the homotopy argument. As in [14, p 246], define

$$\phi_\tau(x) = \min\{\phi(x), \phi^*(x - \tau)\}, \quad \tau \in \mathbb{R}, x \in \mathbb{R}. \tag{45}$$

The minimum is calculated componentwise. For each τ, ϕ_τ is clearly continuous and increasing. It also follows directly from (45) that for each fixed $x \in \mathbb{R}$, $\phi_\tau(x)$ is a decreasing function of τ. The following crucial property of ϕ_τ is proved in [14, p 246];

$$\phi_{-2\eta_2}(x) = \phi(x) \ \text{ and } \ \phi_{2\eta_2}(x) = \phi^*(x - 2\eta_2) \text{ for all } x \in \mathbb{R}. \tag{46}$$

The existence theory for the initial-value problem for (6) in the Appendix requires the initial data in $\mathfrak{C}^1$. We introduce mollifications of ϕ_τ in order to consider τ-dependent initial-value problems. For $b \in (0, 1)$, let $\kappa_b : \mathbb{R} \to [0, \infty)$ be a standard normalised mollifier, supported in $[-b, b]$ (see, for example, [6, p 46]). For $\tau \in \mathbb{R}$, $b \in (0, 1), x \in \mathbb{R}$, let

$$\psi_{\tau,b}(x) = (\phi_\tau * \kappa_b)(x) = \int_{-\infty}^{\infty} \phi_\tau(x - s)\kappa_b(s)\ ds. \tag{47}$$

By construction, $E^- \leq \psi_{\tau,b}(x) \leq E^+$ for all x. It follows from Theorem A.7 (Appendix) that the initial-value problem (6) with initial data $\psi_{\tau,b}$ has a unique classical solution $v^{\psi_{\tau,b}}$ that exists for all time, and that

$$E^- \leq v^{\psi_{\tau,b}}(x, t) \leq E^+ \quad \text{for all} \ \ x \in \mathbb{R}, \quad t \geq 0. \tag{48}$$

The approach is to advance the parameter τ with step $-h < 0$ (to be determined) from $\tau = 2\eta_2$ to $\tau = -2\eta_2$, at each stage proving that the solution $v^{\psi_{\tau,b}}$ with initial data $\psi_{\tau,b}$ converges in $\mathfrak{C}^1$ to a translate of w. At $\tau = -2\eta_2$, the initial data is $\phi * \kappa_b$, by (46); letting $b \to 0$ will then yield the required result.

We seek $h_\epsilon > 0$, independent of $b \in (0, 1), \tau \in \mathbb{R}, T \geq 1$, such that

$$\|v^{\psi_{\tau-h_\epsilon,b}}(\cdot, T) - v^{\psi_{\tau,b}}(\cdot, T)\|_{\mathfrak{C}^1} \leq \frac{\nu_\epsilon}{4}. \tag{49}$$

By Landau's inequality,

$$\|(v^{\psi_{\tau-h,b}} - v^{\psi_{\tau,b}})_x(\cdot, T)\|_{\mathfrak{C}} \leq 2\|(v^{\psi_{\tau-h,b}} - v^{\psi_{\tau,b}})(\cdot, T)\|_{\mathfrak{C}}^{\frac{1}{2}}\|(v^{\psi_{\tau-h,b}} - v^{\psi_{\tau,b}})_{xx}(\cdot, T)\|_{\mathfrak{C}}^{\frac{1}{2}} \tag{50}$$

for each $b \in (0,1), \tau \in \mathbb{R}, T \geq 1$ and $h > 0$. We now show that the first factor on the right of (50) is small when h is small. Note first that for $x \in \mathbb{R}, \tau \in \mathbb{R}, h > 0$,

$$\phi_\tau(x) \leq \phi_{\tau-h}(x) \leq \phi_\tau(x+h). \tag{51}$$

Since mollification preserves ordering and commutes with translation, it follows that for $b \in (0,1)$,

$$\psi_{\tau,b}(x) \leq \psi_{\tau-h,b}(x) \leq \psi_{\tau,b}(x+h). \tag{52}$$

Now since f satisfies **(f1) - (f2)**, the comparison principle Theorem A.2 (Appendix) yields that

$$v^{\psi_{\tau,b}}(x,t) \leq v^{\psi_{\tau-h,b}}(x,t) \leq v^{\psi_{\tau,b}}(x+h,t), \quad x \in \mathbb{R}, t > 0. \tag{53}$$

So by the Mean Value Inequality, for $t > 0, x \in \mathbb{R}$,

$$\|v^{\psi_{\tau-h,b}}(x,t) - v^{\psi_{\tau,b}}(x,t)\| \leq \|v^{\psi_{\tau,b}}(x+h,t) - v^{\psi_{\tau,b}}(x,t)\| \leq h\|(v^{\psi_{\tau,b}})_x(\cdot,t)\|_{\mathfrak{C}}. \tag{54}$$

By Theorem A.8 (Appendix) there exists $K_1 > 0$, independent of $t \geq 1, \tau \in \mathbb{R}, b \in (0,1)$, such $\|(v^{\psi_{\tau,b}})_x(\cdot,t)\|_{\mathfrak{C}} \leq K_1$. Hence for each $h > 0$, $t \geq 1, \tau \in \mathbb{R}, b \in (0,1)$,

$$\|v^{\psi_{\tau-h,b}}(\cdot,t) - v^{\psi_{\tau,b}}(\cdot,t)\|_{\mathfrak{C}} \leq K_1 h. \tag{55}$$

It follows from (48) and Theorem A.8 that the second factor on the right of (50) is bounded independently of $\tau \in \mathbb{R}, h > 0, b \in (0,1), T \geq 1$. The existence of $h_\epsilon > 0$ satisfying (49), independent of $b \in (0,1), \tau \in \mathbb{R}$ and $T \geq 1$, thus follows from (50) and (55). We choose $h_\epsilon > 0$ smaller if necessary so that there exists $n \in \mathbb{N}$ such that

$$4\eta_2 = nh_\epsilon. \tag{56}$$

Now $\|\phi^* * \kappa_b - w\|_{\mathfrak{C}^1} \to 0$ as $b \to 0$, so it follows from (44) that for $b \in (0, b_0)$ say, $\|\phi^* * \kappa_b - w\|_{\mathfrak{C}^1} < \nu_\epsilon$. Hence $\|\psi_{2\eta_2,b} - w(\cdot - 2\eta_2)\|_{\mathfrak{C}^1} < \nu_\epsilon$. With $\gamma_\epsilon, K_\epsilon, \delta_\epsilon > 0$ (independent of b) as in Theorem 3.1, there exists $\chi_{2\eta_2,b} \in [-2\eta_2 - \delta_\epsilon, -2\eta_2 + \delta_\epsilon]$ such that

$$\|v^{\psi_{2\eta_2,b}}(\cdot,t) - w(\cdot + \chi_{2\eta_2,b})\|_{\mathfrak{C}^1} \leq K_\epsilon e^{-\gamma_\epsilon t} \quad \text{for all} \quad t > 0. \tag{57}$$

Next define

$$T_\epsilon = \max\{1, \frac{1}{\gamma_\epsilon} \log \frac{4K_\epsilon}{\nu_\epsilon}\}. \tag{58}$$

(Clearly T_ϵ is independent of $b \in (0, b_0)$.) So by (57) and (58),

$$\|v^{\psi_{2\eta_2,b}}(\cdot,T_\epsilon) - w(\cdot + \chi_{2\eta_2,b})\|_{\mathfrak{C}^1} \leq \frac{\nu_\epsilon}{4}. \tag{59}$$

Together with (49), this yields that

$$\|v^{\psi_{2\eta_2-h_\epsilon,b}}(\cdot,T_\epsilon) - w(\cdot + \chi_{2\eta_2,b})\|_{\mathfrak{C}^1} \leq \frac{\nu_\epsilon}{2}. \tag{60}$$

So by Theorem 3.1, there exists $\chi_{2\eta_2-h_\epsilon,b} \in [\chi_{2\eta_2,b} - \delta_\epsilon, \chi_{2\eta_2,b} + \delta_\epsilon] \subset [-2\eta_2 - 2\delta_\epsilon, -2\eta_2 + 2\delta_\epsilon]$ such that for $t > T_\epsilon$,

$$\|v^{\psi_{2\eta_2-h_\epsilon,b}}(\cdot,t) - w(\cdot + \chi_{2\eta_2-h_\epsilon,b})\|_{\mathfrak{C}^1} \leq K_\epsilon e^{-\gamma_\epsilon(t-T_\epsilon)}. \tag{61}$$

Arguing by induction, it follows that given $m \in \mathbb{N}$, there exist $\chi_{2\eta_2-kh_\epsilon,b} \in [-2\eta_2 - k\delta_\epsilon, -2\eta_2 + k\delta_\epsilon]$ for each $0 \leq k \leq m$ such that

$$\|v^{\psi_{2\eta_2-mh_\epsilon,b}}(\cdot,mT_\epsilon) - w(\cdot + \chi_{2\eta_2-(m-1)h_\epsilon,b})\|_{\mathfrak{C}^1} \leq \frac{\nu_\epsilon}{2}, \tag{62}$$

and for $t > mT_\epsilon$,

$$\|v^{\psi_{2\eta_2 - mh_\epsilon,b}}(\cdot,t) - w(\cdot + \chi_{2\eta_2 - mh_\epsilon,b})\|_{\mathfrak{C}^1} \leq K_\epsilon e^{-\gamma_\epsilon(t-mT_\epsilon)}. \tag{63}$$

In particular, (62) and (63) hold for n satisfying (56). Since $\psi_{-2\eta_2,b} = \phi * \kappa_b$, this yields that for each $b \in (0,b_0)$, there exists $\chi_{-2\eta_2+h_\epsilon,b} \in [-2\eta_2 - (n-1)\delta_\epsilon, -2\eta_2 + (n-1)\delta_\epsilon]$ such that

$$\|v^{\phi*\kappa_b}(\cdot,nT_\epsilon) - w(\cdot + \chi_{-2\eta_2+h_\epsilon,b})\|_{\mathfrak{C}^1} \leq \frac{\nu_\epsilon}{2}. \tag{64}$$

We now let $b \to 0$. The interval $[-2\eta_2 - (n-1)\delta_\epsilon, -2\eta_2 + (n-1)\delta_\epsilon]$ is independent of $b \in (0,b_0)$. So there is a sequence $\{b_k\} \subset (0,b_0), b_k \downarrow 0$ and $\chi_\epsilon \in [-2\eta_2 - (n-1)\delta_\epsilon, -2\eta_2 + (n-1)\delta_\epsilon]$ such that

$$\chi_{-2\eta_2+h_\epsilon,b_k} \to \chi_\epsilon \quad \text{as} \quad k \to \infty. \tag{65}$$

Thus there exists $k_0 \in \mathbb{N}$ such that

$$k \geq k_0 \Rightarrow \|w(\cdot + \chi_{-2\eta_2+h_\epsilon,b_k}) - w(\cdot + \chi_\epsilon)\|_{\mathfrak{C}^1} \leq \frac{\nu_\epsilon}{4}. \tag{66}$$

Proposition A.3 (Appendix) yields the existence of $r, K > 0$ such that for n as in (56) and $\hat{\phi}, \tilde{\phi} \in \mathfrak{C}^1$,

$$\|\hat{\phi} - \tilde{\phi}\|_{\mathfrak{C}^1} \leq r \Rightarrow \|v^{\hat{\phi}}(\cdot,nT_\epsilon) - v^{\tilde{\phi}}(\cdot,nT_\epsilon)\|_{\mathfrak{C}^1} \leq K\|\hat{\phi} - \tilde{\phi}\|_{\mathfrak{C}^1}. \tag{67}$$

Hence since $\|\phi - \phi * \kappa_b\|_{\mathfrak{C}^1} \to 0$ as $s \to 0$, there exists $k_1 \in \mathbb{N}$ such that

$$k \geq k_1 \Rightarrow \|v^{\phi}(\cdot,nT_\epsilon) - v^{\phi*\kappa_{b_k}}(\cdot,nT_\epsilon)\|_{\mathfrak{C}^1} \leq \frac{\nu_\epsilon}{4}. \tag{68}$$

So by (64), (66) and (68),

$$\|v^{\phi}(\cdot,nT_\epsilon) - w(\cdot + \chi_\epsilon)\|_{\mathfrak{C}^1} \leq \nu_\epsilon. \tag{69}$$

Theorem 3.1 yields that

$$\|v^{\phi}(\cdot,t) - w(\cdot + \chi_\epsilon)\|_{\mathfrak{C}^1} \leq K_\epsilon e^{-\gamma_\epsilon(t-nT_\epsilon)} \quad \text{for} \quad t > nT_\epsilon. \tag{70}$$

Since v^ϕ is independent of ϵ, and w is not periodic, it is immediate that $\chi_{\epsilon_1} = \chi_{\epsilon_2}$ for any $\epsilon_1, \epsilon_2 \in (0,\gamma_0)$. The result follows. □

5 Global stability for general initial data

We will invoke an idea from [13]. First a preliminary lemma, which is a modification of [13, Lemma 3.3]. This result will be used later, in the proof of Theorem 5.3, as part of an argument by contradiction.

Lemma 5.1 *Let $D, G : \mathbb{R} \times [0,\infty) \to M^{N\times N}$ be continuous $N \times N$-matrix-valued-functions, uniformly bounded on $\mathbb{R} \times [0,\infty)$, such that $D(x,t)$ is diagonal and the off-diagonal elements of $G(x,t)$ are non-negative for each $(x,t) \in \mathbb{R} \times [0,\infty)$. Let $\mathfrak{h}$ be a non-negative, uniformly bounded solution of*

$$\mathfrak{h}_t(x,t) = A\mathfrak{h}_{xx}(x,t) + D(x,t)\mathfrak{h}_x(x,t) + G(x,t)\mathfrak{h}(x,t), \quad (x,t) \in \mathbb{R} \times [0,\infty), \tag{71}$$

such that $\mathfrak{h}_t$ is uniformly bounded for $t \geq \frac{1}{2}$ and there exist $\mu_0, M_0 > 0$ such that for each $t \geq 0$,

$$\sup_{x\in\mathbb{R}} \left(\min_{1\leq i\leq N} \mathfrak{h}_i(x,t) \right) = \max_{|x|\leq M_0} \left(\min_{1\leq i\leq N} \mathfrak{h}_i(x,t) \right) \geq \mu_0. \tag{72}$$

Then for each $M \geq M_0$, there exists $\alpha(M) > 0$ such that for all $t \geq 1$,

$$\min_{|x|\leq M} \min_{1\leq i\leq N} \mathfrak{h}_i(x,t) \geq \alpha(M). \tag{73}$$

Proof. Let $M \geq M_0$ and recall that $\mathfrak{e} = (1, \ldots, 1)$. It follows from (72) that for each $T \geq 0$, there exists $x^T \in [-M_0, M_0]$ such that $\mathfrak{h}(x^T, T) \geq \mu_0 \mathfrak{e}$. Furthermore, $\mathfrak{h}_t(x,t)$ is bounded independently of $x \in \mathbb{R}, t \geq \frac{1}{2}$, so there exists $T_0 \in (0, \frac{1}{2})$, independent of $T \geq 1$, such that

$$T \geq 1, |\hat{t}| \leq T_0 \Rightarrow \mathfrak{h}(x^T, T + \hat{t}) \geq \frac{\mu_0}{2}\mathfrak{e}. \tag{74}$$

We will construct a strictly positive function which lies beneath $\mathfrak{h}(x,t)$ for all $t \geq 1$. By the hypotheses on D and G, there are constant diagonal matrices D^-, D^+ and a constant negative-definite diagonal matrix G^- such that

$$G^-_{ij} \leq G_{ij}(x,t) \quad \text{and} \quad D^-_{ij} \leq D_{ij}(x,t) \leq D^+_{ij} \quad \text{for all} \quad x \in \mathbb{R}, t \geq 0, i, j \in \{1, \ldots, N\}. \tag{75}$$

Consider the two initial-boundary-value problems for $u^+ : [0,\infty)\times[0, 2T_0] \to \mathbb{R}^N$ and $u^- : (-\infty, 0]\times[0, 2T_0] \to \mathbb{R}^N$;

$$\begin{aligned}
&u^\pm_t = Au^\pm_{xx} + D^\pm u^\pm_x + G^- u^\pm, \quad (\pm x, t) \in (0,\infty) \times (0, 2T_0),\\
&u^\pm(0,t) = \tfrac{\mu_0}{2}\mathfrak{e} \text{ for } t \in [0, 2T_0],\\
&u^\pm(x, 0) = 0 \text{ for } \pm x \in (0,\infty),\\
&u^\pm(x,t) \to 0 \text{ as } \pm x \to \infty.
\end{aligned}$$

Since $A, D^\pm, G^-$ are diagonal, we can solve these explicitly using Laplace transforms to find that for each $i \in \{1, \ldots, N\}$ and $(\pm x, t) \in (0,\infty) \times [0, 2T_0]$,

$$u^\pm_i(x,t) = \pm \left(\frac{A_i}{4\pi}\right)^{\frac{1}{2}} x e^{-\frac{1}{2}D^\pm_i x} \int_0^t s^{-\frac{3}{2}} \exp\left[-\left(\frac{(D^\pm_i)^2}{A_i} - G^-_i \right) s - \frac{A_i x^2}{4s} \right] ds. \tag{76}$$

We will show that $u^+_x(x,t) < 0$ for all $x > 0, t > 0$. (76) yields that for each $i \in \{1, \ldots, N\}$,

$$\left(u^+_i\right)_x(x,t) = \left(\frac{A_i}{4\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}D^+_i x} \int_0^t \left\{ 1 - \frac{D^+_i x}{2} - \frac{A_i x^2}{2s} \right\} s^{-\frac{3}{2}} \exp\left[-\left(\frac{(D^+_i)^2}{A_i} - G^-_i \right) s - \frac{A_i x^2}{4s} \right] ds.$$

Fix $t \in (0, 2T_0]$ and let

$$x^t_+ = \inf\{x > 0 : u^+_x(s,t) < 0 \ \text{ for each } \ s \in [x,\infty)\}.$$

The formula for u^+_x shows that $u^+_x(x,t) < 0$ for x sufficiently large. So $x^t_+ \in [0,\infty)$. Suppose that $x^t_+ > 0$. Then for some $i \in \{1, \ldots, N\}$, $(u^+_i)_x(x^t_+, t) = 0$. So

$$A_i(u^+_i)_{xx}(x^t_+, t) = (u^+_i)_t(x^t_+, t) - G^-_{ii} u^+_i(x^t_+, t).$$

Now $u^+_i > 0$, $G^-_{ii} < 0$ and it is clear from (76) that $(u^+_i)_t > 0$. So since $A_i > 0$, $(u^+_i)_{xx}(x^t_+, t) > 0$. But this implies that u^+_i has a strict local minimum at x^t_+, which contradicts the fact that

$(u_i^+)_x(x,t) < 0$ for all $x > x_+^t$. Whence $x_+^t = 0$. A similar argument shows that $u_x^-(x,t) > 0$ for each $t > 0, x < 0$.

Fix $T \geq 1$. Let $u^{T,+} : [x^T, \infty) \times [T - T_0, T + T_0] \to \mathbb{R}^N$, $u^{T,-} : (-\infty, x^T] \times [T - T_0, T + T_0] \to \mathbb{R}^N$ denote the unique solutions of the two initial-boundary-value problems

$$\begin{aligned} &u_t^{T,\pm} = Au_{xx}^{T,\pm} + D^{\pm}u_x^{T,\pm} + G^- u^{T,\pm}, \quad (\pm\{x - x^T\}, t) \in (0,\infty) \times (T - T_0, T + T_0), \\ &u^{T,\pm}(x^T, t) = \tfrac{\mu_0}{2}\mathfrak{e} \text{ for } t \in [T - T_0, T + T_0], \\ &u^{T,\pm}(x, T - T_0) = 0 \text{ for } \pm\{x - x^T\} \in (0,\infty), \\ &u^{T,\pm}(x,t) \to 0 \text{ as } \pm x \to \infty. \end{aligned} \tag{77}$$

Clearly,

$$u^{T,\pm}(x,t) = u^{\pm}(x - x^T, t - T + T_0), \quad (\pm\{x - x^T]\}, t) \in [0,\infty) \times [T - T_0, T + T_0].$$

So, since $\pm u_x^{\pm} < 0$ for $t, \pm x > 0$,

$$\min_{x\in[x^T,M]} u^{T,+}(x,T) \geq \min_{x\in[0,M_0+M]} u^+(x,T_0) = u^+(M_0 + M, T_0), \tag{78}$$

$$\min_{x\in[-M,x^T]} u^{T,-}(x,T) \geq \min_{x\in[-M_0-M,0]} u^-(x,T_0) = u^-(-M_0 - M, T_0). \tag{79}$$

Now $u^{T,\pm}(x,t) > 0, \pm u_x^{T,\pm}(x,t) < 0$ for $(\pm\{x - x^T\}, t) \in (0,\infty) \times (T - T_0, T + T_0)$, so it follows from (75) and (77) that for such (x,t),

$$u_t^{T,\pm}(x,t) - Au_{xx}^{T,\pm}(x,t) - D(x,t)u_x^{T,\pm}(x,t) - G(x,t)u^{T,\pm}(x,t) \leq 0. \tag{80}$$

So since (74) holds and $\mathfrak{h}$ is non-negative, it follows from the positivity theorem Theorem A.1 (i) (Appendix) that

$$\mathfrak{h}(x,t) \geq u^{T,\pm}(x,t), \quad (\pm\{x - x^T\}, t) \in [0,\infty) \times [T - T_0, T + T_0]. \tag{81}$$

Hence by (78), (79), (81),

$$\min_{x\in[-M,M]} \mathfrak{h}(x,T) \geq \min\{u^0(M_0 + M), T_0, v^0(-M_0 - M, T_0)\}. \tag{82}$$

The right-hand side of (82) is a strictly positive number independent of $x \in [-M,M], T \geq 1$. The result follows. $\square$

For $\phi \in \mathfrak{C}^1$, define its omega limit set

$$W(\phi) = \{\psi \in \mathfrak{C}^1 : \text{ there is a sequence } t_n \to \infty \text{ such that } \|v^\phi(\cdot, t_n) - \psi\|_{\mathfrak{C}^1} \to 0\}. \tag{83}$$

Theorem A.6 (Appendix) gives conditions on the initial data ϕ under which wave-dependent sub- and super-solutions for (6) can be constructed. This yields important information about $W(\phi)$.

Lemma 5.2 *Let $\hat{\eta} > 0$ be as in Theorem A.6 (Appendix), and let $\phi \in \mathfrak{C}^1$ satisfy (110), (111) for some $\eta \in (0, \hat{\eta})$. Then*

(i) $W(\phi)$ is nonempty and compact;

(ii) there exists $\hat{x}(\phi) \in \mathbb{R}$ such that for all $x \in \mathbb{R}, \psi \in W(\phi)$,

$$w(x - \hat{x}(\phi)) \leq \psi(x) \leq w(x + \hat{x}(\phi));$$

(iii) $(\psi)'(x) \to 0$ as $|x| \to \infty$ for each $\psi \in W(\phi)$;

(iv) if $\psi \in W(\phi)$, then $v^{\psi}(\cdot, t) \in W(\phi)$ for all $t \geq 0$ and $W(\psi) \subset W(\phi)$.

Proof. The *a priori* estimates of Theorem A.8 (Appendix), the Arzela-Ascoli theorem and estimate (112) of Theorem A.6 (Appendix) together show (i). Estimate (112) also yields (ii). (iii) follows from (ii), Theorem A.8 and Landau's inequality on a half-line. (iv) is a consequence of definition (83), the last part of Proposition A.3 (Appendix) and the semigroup property of solutions of (6). □

The next theorem is the key. We include a proof for completeness; the approach is a minor modification of [13, Lemma 3.4].

Theorem 5.3 *Let $\phi \in \mathfrak{C}^1$ be as in Lemma 5.2. Then there exists $\psi_0 \in W(\phi)$, $\psi_0(x) \to E^{\pm}$ as $x \to \pm\infty$, $(\psi_0)'(x) \to 0$ as $|x| \to \infty$ and $(\psi_0)'(x) \geq 0$ for each $x \in \mathbb{R}$.*

Proof. Define $\mathcal{F} : W(\phi) \to [0, \infty]$ by

$$\mathcal{F}(\psi) = \inf\{\chi_0 > 0 : \psi(x + \chi) \geq \psi(x) \text{ for all } \chi \geq \chi_0, x \in \mathbb{R}\}. \tag{84}$$

Note that since $W(\phi) \subset \mathfrak{C}$, $\psi(x + \mathcal{F}(\psi)) \geq \psi(x)$ for each $x \in \mathbb{R}$, $\psi \in W(\phi)$. Lemma 5.2 (ii) shows that $\mathcal{F}(\psi) < \infty$ for each $\psi \in W(\phi)$. It follows from Lemma 5.2 (i) that $\mathcal{F}$ attains its minimum $\mathcal{F}_0$ at a point $\psi_0 \in W(\phi)$. Lemma 5.2 (ii), (iii) ensure that $\psi_0(x) \to E^{\pm}$ as $x \to \pm\infty$ and $(\psi_0)'(x) \to 0$ as $|x| \to \infty$.

If $\mathcal{F}_0 = 0$, then $(\psi_0)'(x) \geq 0$ for each $x \in \mathbb{R}$. So suppose, for contradiction, that $\mathcal{F}_0 > 0$. We consider the solution v^{ψ_0} of (6) with initial data ψ_0. Note first that Lemma 5.2 (iv) states that $v^{\psi_0}(\cdot, t) \in W(\phi)$ for all $t \geq 0$. By the choice of ψ_0 as the minimiser of $\mathcal{F}$ and Theorem A.2 (Appendix), $\mathcal{F}(v^{\psi_0}(\cdot, t)) = \mathcal{F}_0$ for all $t \geq 0$, so

$$v^{\psi_0}(x + \mathcal{F}_0, t) \geq v^{\psi_0}(x, t) \text{ for all } x \in \mathbb{R}, t \geq 0.$$

In fact, since $\mathcal{F}_0 > 0$ and Lemma 5.2 (ii) holds, there exist $\mu_0 > 0, M_0 > 0$ such that for all $t \geq 0$,

$$\sup_{x \in \mathbb{R}} \left(\min_{1 \leq i \leq N} v_i^{\psi_0}(x + \mathcal{F}_0, t) - v_i^{\psi_0}(x, t) \right) = \max_{|x| \leq M_0} \left(\min_{1 \leq i \leq N} v_i^{\psi_0}(x + \mathcal{F}_0, t) - v_i^{\psi_0}(x, t) \right) \geq \mu_0. \tag{85}$$

Let $q_0, \mathfrak{e}^{\pm}, \nu$ be as in the preamble to Theorem A.5 (Appendix). By Theorem A.8 (Appendix), $\|v_{xx}^{\psi_0}(\cdot, t)\|_{\mathfrak{C}}$ is bounded independently of $t \geq 1$. So it follows from Lemma 5.2 (ii) and Landau's inequality on a half line that $v_x^{\psi_0}(x, t) \to 0$ as $|x| \to \infty$ at a rate independent of $t \geq 1$. Thus Lemma 5.2 (ii) and (101) give that there exists $M \geq M_0$ such that for all $t \geq 0$, $\sigma \in [0, 1]$ and each $\mathcal{F} \in [0, \mathcal{F}_0]$,

$$\begin{aligned} \pm x \quad &\geq \quad M \Rightarrow \\ & d_q f[\sigma v^{\psi_0}(x + \mathcal{F}, t) + (1 - \sigma) v^{\psi_0}(x, t), \sigma v_x^{\psi_0}(x + \mathcal{F}, t) + (1 - \sigma) v_x^{\psi_0}(x, t)] \mathfrak{e}^{\pm} \leq -\frac{\nu}{2} \mathfrak{e}^{\pm}. \end{aligned} \tag{86}$$

For $\delta \geq 0$, define

$$\mathfrak{h}^\delta(x,t) = v^{\psi_0}(x+\mathcal{F}_0-\delta,t) - v^{\psi_0}(x,t), \quad x\in\mathbb{R}, t\geq 0. \tag{87}$$

Then $\mathfrak{h}^0 \geq 0$, and for each $\delta \geq 0$, $\mathfrak{h}^\delta$ is a solution of

$$\mathfrak{h}^\delta_t(x,t) = A\mathfrak{h}^\delta_{xx}(x,t) + c\mathfrak{h}^\delta_x(x,t) + D^\delta(x,t)\mathfrak{h}^\delta_x(x,t) + G^\delta(x,t)\mathfrak{h}^\delta(x,t), \tag{88}$$

where

$$D^\delta(x,t) = \int_0^1 d_p f[\sigma v^{\psi_0}(x+\mathcal{F}_0-\delta,t) + (1-\sigma)v^{\psi_0}(x,t), \sigma v^{\psi_0}_x(x+\mathcal{F}_0-\delta,t) + (1-\sigma)v^{\psi_0}_x(x,t)]\, d\sigma,$$

$$G^\delta(x,t) = \int_0^1 d_q f[\sigma v^{\psi_0}(x+\mathcal{F}_0-\delta,t) + (1-\sigma)v^{\psi_0}(x,t), \sigma v^{\psi_0}_x(x+\mathcal{F}_0-\delta,t) + (1-\sigma)v^{\psi_0}_x(x,t)]\, d\sigma.$$

Since f satisfies (f1) and (f2), the matrices $cI + D^0, G^0$ satisfy the hypotheses on D, G respectively in Lemma 5.1. Also, Theorem A.8 (Appendix) shows that $\mathfrak{h}^0_t(x,t)$ is bounded independently of $x\in\mathbb{R}, t\geq \frac{1}{2}$. So with M as in (86), Lemma 5.1 (applied to the function $\mathfrak{h}^0$) together with (85) imply the existence of $\alpha(M) > 0$ such that for each $t \geq 1$,

$$|x| \leq M \Rightarrow \mathfrak{h}^0(x,t) \geq \alpha(M). \tag{89}$$

Theorem A.8 shows that $(v^{\psi_0})_x(x,t)$ is bounded independently of $x\in\mathbb{R}, t\geq 1$. So there exists $\delta(M)\in(0,\mathcal{F}_0)$ such that for each $t\geq 1$,

$$|x| \leq M, \delta\in[0,\delta(M)] \Rightarrow \mathfrak{h}^\delta(x,t) \geq \tfrac{1}{2}\alpha(M). \tag{90}$$

Now by Lemma 5.2 (i) and (iv), there is a sequence $t_n\to\infty$ and $\psi_1 \in W(\psi_0) \subset W(\phi)$ such that

$$\|v^{\psi_0}(\cdot,t_n) - \psi_1\|_{\mathfrak{C}^1} \to 0 \quad \text{as} \quad n\to\infty. \tag{91}$$

Fix $\delta\in[0,\delta(M)]$. (90) and (91) show that $\psi_1(x+\mathcal{F}_0-\delta) \geq \psi_1(x)$ for all $x\in[-M,M]$. Consider $x\geq M$. Now $\mathfrak{h}^\delta$ is uniformly bounded, $\mathfrak{h}^\delta(M,t) \geq \frac{1}{2}\alpha(M)$ for $t\geq 1$ and (86) holds. So Theorem A.1 (i) (Appendix) (applied to (88)) shows that there is a constant $K^\delta > 0$ such that for all $x\geq M, t\geq 1$,

$$\mathfrak{h}^\delta(x,t) \geq -K^\delta e^{-\frac{\nu}{2}t}\mathfrak{e}^+. \tag{92}$$

Whence $\psi_1(x+\mathcal{F}_0-\delta) \geq \psi_1(x)$ for each $x\geq M$. Similarly, $\psi_1(x+\mathcal{F}_0-\delta)\geq\psi_1(x)$ for $x\leq -M$. So

$$\psi_1(x+\mathcal{F}_0-\delta) \geq \psi_1(x) \quad \text{for all} \quad x\in\mathbb{R}, \ \delta\in[0,\delta(M)]. \tag{93}$$

But it follows from (91) and the fact that $\mathcal{F}(v^{\psi_0}(\cdot,t)) = \mathcal{F}_0$ for all $t\geq 0$ that $\mathcal{F}(\psi_1) = \mathcal{F}_0$. This contradicts (93). Thus $\mathcal{F}_0 = 0$ and the result follows. □

The main result of this paper is the following.

Theorem 5.4 *Let $f\in C^1(\mathbb{R}^N\times\mathbb{R}^N,\mathbb{R}^N)$ satisfy* **(f1) - (f4)***. Let $\hat\eta > 0$ be as in Theorem A.6, and let ϕ satisfy (110), (111) for some $\eta\in(0,\hat\eta)$. Then there exists $\chi_\infty\in\mathbb{R}$ such that for each $\epsilon\in(0,\gamma_0)$, there exists $N_\epsilon > 0$ such that the solution v^ϕ of (6) with initial data ϕ satisfies*

$$\|v^\phi(\cdot,t) - w(\cdot+\chi_\infty)\|_{\mathfrak{C}^1} \leq N_\epsilon e^{-\gamma_\epsilon t}, \quad \textit{for all} \quad t>0. \tag{94}$$

Proof. Theorem 5.3, Theorem 4.1 and Lemma 5.2 (iv) show that there exists $\chi_\infty \in \mathbb{R}$ such that $w(\cdot + \chi_\infty) \in W(\phi)$. The result then follows from Theorem 3.1. □

This implies a uniqueness result for travelling-wave solutions of (3).

Corollary 5.5 *Let $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ satisfy* **(f1)-(f4)**. *Let $\hat{\eta} > 0$ be as in Theorem A.6, and let $\phi \in \mathfrak{C}^1$ satisfy (110), (111) for some $\eta \in (0, \hat{\eta})$. Suppose that there exists $\hat{c} \in \mathbb{R}$ such that $u(x,t) := \phi(x - \hat{c}t)$ is a travelling-wave solution of (3). Then $\hat{c} = c$ and there exists $\chi_\infty \in \mathbb{R}$ such that $\phi(\cdot) = w(\cdot + \chi_\infty)$. (Here w, c are as in* **(TW)**.*)*

Proof. Theorem 5.4 shows that there exists $\chi_\infty \in \mathbb{R}$ such that

$$\|u(\cdot + ct, t) - w(\cdot + \chi_\infty)\|_{\mathfrak{C}^1} = \|\phi(\cdot + \{c - \hat{c}\}t) - w(\cdot + \chi_\infty)\|_{\mathfrak{C}^1} \to 0 \quad \text{as} \quad t \to \infty. \tag{95}$$

Suppose that $c > \hat{c}$. Since $w(x) \to E^-$ as $x \to -\infty$, we can choose $\hat{x} \in \mathbb{R}$ such that $w(\hat{x} + \chi_\infty) < E^+ - \hat{\eta}\mathfrak{e}^+$. But since ϕ satisfies (111) and $c - \hat{c} > 0$, $\phi(\hat{x} + \{c - \hat{c}\}t) > E^+ - \hat{\eta}\mathfrak{e}^+$ for t sufficiently large. This contradicts (95), so $c \le \hat{c}$. A similar argument shows that $c \ge \hat{c}$. Whence $c = \hat{c}$. The result now follows from (95). □

A Appendix

Comparison theorem

For $T > 0$, define

$$\Gamma_T = \{v \in C(\mathbb{R} \times [0,T], \mathbb{R}^N) : v_t, v_x, v_{xx} \text{ are continuous on } \mathbb{R} \times (0,T)\},$$

$$\Gamma_T^+ = \{v \in C([0,\infty) \times [0,T], \mathbb{R}^N) : v_t, v_x, v_{xx} \text{ are continuous on } (0,\infty) \times (0,T)\}.$$

For $v \in \Gamma_T$, $(x,t) \in \mathbb{R} \times (0,T]$ (or $v \in \Gamma_T^+$, $(x,t) \in (0,\infty) \times (0,T)$), define

$$\mathcal{M}(v)(x,t) = -v_t(x,t) + Av_{xx}(x,t) + D(x,t)v_x(x,t) + G(x,t)v(x,t), \tag{96}$$

and

$$\mathcal{N}(v)(x,t) = -v_t(x,t) + Av_{xx}(x,t) + cv_x(x,t) + f(v(x,t), v_x(x,t)), \tag{97}$$

where A satisfies **(a)**, $c \in \mathbb{R}$, $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ satisfies **(f1) - (f2)** and $D, G : \mathbb{R} \times [0,T] \to M^{N \times N}$ are continuous $N \times N$ matrix-valued functions, bounded on $\mathbb{R} \times [0,T]$, such that D is diagonal and the off-diagonal elements of G are non-negative. [14, p 241, Lemma 5.2 and p 242, Theorem 5.3] yield the following positivity results.

Theorem A.1 *(i) Let $v \in \Gamma_T^+$ be such that v is bounded on $[0,\infty) \times [0,T]$ and $\mathcal{M}(v)(x,t) \le 0$ for $(x,t) \in (0,\infty) \times (0,T]$. If $v(x,0) \ge 0$ for all $x \in \mathbb{R}$ and $v(0,t) \ge 0$ for each $t \in [0,T]$, then $v(x,t) \ge 0$ for all $(x,t) \in [0,\infty) \times [0,T]$.*

(ii) Let $v \in \Gamma_T$ be such that v is bounded on $\mathbb{R} \times [0,T]$ and $\mathcal{M}(v)(x,t) \le 0$ for $(x,t) \in \mathbb{R} \times (0,T]$. If $v(x,0) \ge 0$ for all $x \in \mathbb{R}$, then $v(x,t) \ge 0$ for all $(x,t) \in \mathbb{R} \times [0,T]$.

The following comparison principle for (6) is a straightforward consequence of Theorem A.1 (ii).

Theorem A.2 *Let $v, \tilde{v} \in \Gamma_T$ be such that $v, \tilde{v}, v_x, \tilde{v}_x$ are bounded on $\mathbb{R} \times (0, T]$, $\mathcal{N}(\tilde{v})(x,t) \leq 0$ and $\mathcal{N}(v)(x,t) \geq 0$ for $(x,t) \in \mathbb{R} \times (0,T]$. Suppose that $\tilde{v}(x,0) - v(x,0) \geq 0$ for all $x \in \mathbb{R}$. Then $\tilde{v}(x,t) \geq v(x,t)$ for all $(x,t) \in \mathbb{R} \times [0,T]$.*

Global existence and *a priori* bounds

The abstract existence theory of [11, p 253-275] applies to the concrete problem

$$v_t = Av_{xx} + cv_x + f(v, v_x), \quad x \in \mathbb{R},\ t > 0,\ v(x,t) \in \mathbb{R}^N, \tag{98}$$

$$v(\cdot, 0) = \phi, \tag{99}$$

where A satisfies **(a)**, $c \in \mathbb{R}$ and $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$.

The local existence of a unique solution of (98), (99) and continuous dependence on the initial data (99) are a consequence of [11, p 258, Theorem 7.1.2, p266, Proposition 7.1.9 and p268, Propostion 7.1.10 and p270, Remark 7.1.12]. $\mathfrak{C}^1$ is a suitable choice of space between $\mathfrak{C}^2$ and $\mathfrak{C}$ for the initial data ϕ – see [11, p 253], the embeddings (19) and (21) and Lemma 2.2. The result is the following.

Proposition A.3 *Let $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ and $\phi \in \mathfrak{C}^1$. Then there exists a maximal $\tau(\phi) \in (0, \infty]$ such that there exists a function $V^\phi \in C^1((0, \tau(\phi)), \mathfrak{C}) \cap C((0, \tau(\phi)), \mathfrak{C}^2) \cap C([0, \tau(\phi)), \mathfrak{C}^1)$ such that v^ϕ defined by $v^\phi(x,t) = V^\phi(t)(x)$ for each $x \in \mathbb{R}$, $t \in [0, \tau(\phi))$ satisfies (98), (99). Moreover, there is a unique function $V^\phi : [0, \tau(\phi)) \to \mathfrak{C}^1$ with these properties.*

In addition, given $0 < T < \tau(\phi)$, there exist $r, K > 0$, depending on ϕ and T, such that if $\tilde{\phi} \in \mathfrak{C}^1$ is such that $\|\phi - \tilde{\phi}\|_{\mathfrak{C}^1} < r$, then $\tau(\tilde{\phi}) \geq T$ and

$$\|v^\phi(\cdot, t) - v^{\tilde{\phi}}(\cdot, t)\|_{\mathfrak{C}^1} \leq K \|\phi - \tilde{\phi}\|_{\mathfrak{C}^1} \quad \textit{for each} \quad 0 \leq t \leq T.$$

Under a growth hypothesis on f, the following global existence result, conditional on an *a priori* bound on $\|v(\cdot, t)\|_{\mathfrak{C}}$, is a consequence of [11, p 266, Proposition 7.1.8, p 268, Proposition 7.1.10 and p 272 Proposition 7.2.2].

Proposition A.4 *Suppose that $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ satisfies the growth condition* **(f4)**. *Let $\phi \in \mathfrak{C}^1$ be such that*

$$\sup_{0 \leq \tilde{t} < \tau(\phi)} \|v^\phi(\cdot, \tilde{t})\|_{\mathfrak{C}} = K < \infty, \tag{100}$$

where v^ϕ and $\tau(\phi)$ are as in Proposition A.3. Then $\tau(\phi) = \infty$.

Sub- and supersolutions

Theorem A.2 enables verification of condition (100) under additional hypotheses on f and ϕ. Suppose that $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ satisfies **(f1) - (f4)**. Let $e_0 = \min_{1 \leq i \leq N}\{E_i^+ - E_i^-\} > 0$. Conditions **(f2)-(f3)** and the Perron-Frobenius Theorem together imply the existence of $\nu^+, \nu^- > 0$ and vectors $\mathfrak{e}^+, \mathfrak{e}^- \in \mathbb{R}^N$, $\mathfrak{e}^\pm > 0$, $\|\mathfrak{e}^\pm\| = 1$ such that $d_q f[E^\pm, 0]\mathfrak{e}^\pm = -\nu^\pm \mathfrak{e}^\pm$. Since $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$, it follows that there exist $p_0, \nu > 0$, $q_0 \in (0, \frac{1}{2}e_0)$, $\eta_0 \in (0, \frac{1}{2}e_0)$ such that

$$\left.\begin{array}{c} q \in \mathbb{R}^N, \|q\| \leq q_0 \\ p \in \mathbb{R}^N, \|p\| \leq p_0 \\ \eta \in (0, \eta_0) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} d_q f[E^\pm + q - \eta\mathfrak{e}^\pm, p]\mathfrak{e}^\pm < -\nu\mathfrak{e}^\pm, \\ d_q f[E^\pm + q + \eta\mathfrak{e}^\pm, p]\mathfrak{e}^\pm < -\nu\mathfrak{e}^\pm, \\ f(E^\pm + q - \eta\mathfrak{e}^\pm, p) - f(E^\pm + q, p) \geq \nu\eta\mathfrak{e}^\pm, \\ f(E^\pm + q + \eta\mathfrak{e}^\pm, p) - f(E^\pm + q, p) \leq \nu\eta\mathfrak{e}^\pm. \end{array}\right. \tag{101}$$

Suppose that **(TW)** holds. Since $w'(x) \to 0$ as $|x| \to \infty$, we can choose q_0, η_0 smaller if necessary to ensure that

$$\|w(x) - E^{\pm}\| < q_0 + \eta_0 \Rightarrow \|w'(x)\| < p_0. \tag{102}$$

Let $\mathfrak{p} \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ be such that $\mathfrak{p}_i(q) = \tilde{\mathfrak{p}}_i(q_i)$ for each $i \in \{1, \dots, N\}$, $q \in \mathbb{R}^N$ (the i-th component of $\mathfrak{p}$ depends only on the i-th component of its argument), where $\tilde{\mathfrak{p}}_i \in C^\infty(\mathbb{R}, \mathbb{R})$ is a smooth monotone function with

$$\tilde{\mathfrak{p}}_i(\omega) = \mathfrak{e}_i^+ \text{ when } |E_i^+ - \omega| \le q_0, \quad \text{and} \quad \tilde{\mathfrak{p}}_i(\omega) = \mathfrak{e}_i^- \text{ when } |E_i^- - \omega| \le q_0. \tag{103}$$

The following construction of sub- and super-solutions is an extension, to nonlinearities f depending on v_x, of constructions in [7] and [12].

Theorem A.5 *There exist $\alpha_0 > 0$ and $\hat{\eta} \in (0, \eta_0]$ such that for any $x_0, x_1 \in \mathbb{R}$ and any $\eta \in [0, \hat{\eta}]$,*

$$\mathcal{N}(\mathbf{s}_{\eta,x_0})(x,t) \ge 0 \quad \textit{and} \quad \mathcal{N}(\mathbf{S}_{\eta,x_1})(x,t) \le 0 \quad \textit{for all} \quad x \in \mathbb{R}, t \ge 0,$$

where for each $i \in \{1, \dots, N\}$,

$$(\mathbf{s}_{\eta,x_0})_i(x,t) = w_i(x - x_0 + \eta\alpha_0 e^{-\nu t}) - \eta e^{-\nu t}\tilde{\mathfrak{p}}_i(w_i(x - x_0 + \eta\alpha_0 e^{-\nu t})) \tag{104}$$

and

$$(\mathbf{S}_{\eta,x_1})_i(x,t) = w_i(x + x_1 - \eta\alpha_0 e^{-\nu t}) + \eta e^{-\nu t}\tilde{\mathfrak{p}}_i(w_i(x + x_1 - \eta\alpha_0 e^{-\nu t})). \tag{105}$$

Here the c in (97) is the velocity of the wave w.

Proof. Let $x_0, x_1 \in \mathbb{R}$ be arbitrary. Let $\alpha_0 > 0$ (to be fixed later), and let $\eta \in (0, \eta_0]$. Define $\mathbf{s}_{\eta,x_0}$ and $\mathbf{S}_{\eta,x_1}$ as in (104) and (105). We will prove the result for $\mathbf{s}_{\eta,x_0}$; the argument for $\mathbf{S}_{\eta,x_1}$ is similar.

Unless otherwise indicated, w, w' are to be evaluated at the point $(x - x_0 + \eta\alpha_0 e^{-\nu t})$. Fix $t \ge 0$. First let x be such that $\|E^+ - w(x - x_0 + \eta\alpha_0 e^{-\nu t})\| \le q_0/2$. For such x, $\tilde{\mathfrak{p}}_i'(w_i(x - x_0 + \eta\alpha_0 e^{-\nu t})) = 0$ and $\tilde{\mathfrak{p}}_i(w_i(x - x_0 + \eta\alpha_0 e^{-\nu t})) = \mathfrak{e}_i^+$ for each $i \in \{1, \dots, N\}$. Hence (101), (102) together with the facts that w is a stationary solution of (6) and that $w'(s) > 0$ for all s yield that

$$\begin{aligned} \mathcal{N}(\mathbf{s}_{\eta,x_0})(x,t) &= \nu\eta\alpha_0 e^{-\nu t} w' - \nu\eta e^{-\nu t}\mathfrak{e}^+ + f(w - \eta e^{-\nu t}\mathfrak{e}^+, w') - f(w, w') \\ &\ge \nu\eta e^{-\nu t}\mathfrak{e}^+ - \nu\eta e^{-\nu t}\mathfrak{e}^+ = 0. \end{aligned}$$

Similarly, $\mathcal{N}(\mathbf{s}_{\eta,x_0})(x,t) \ge 0$ when $\|E^- - w(x - x_0 + \eta\alpha_0 e^{-\nu t})\| \le q_0/2$.

Now let $x \in \mathbb{R}$ be such that

$$\|E^- - w(x - x_0 + \eta\alpha_0 e^{-\nu t})\| \ge q_0/2 \quad \text{and} \quad \|E^+ - w(x - x_0 + \eta\alpha_0 e^{-\nu t})\| \ge q_0/2$$

. Since $w' > 0$, there exists $\beta > 0$, depending only on w and q_0, such that for each $i \in \{1, \dots, N\}$,

$$\|E^- - w(s)\| \ge \frac{q_0}{2} \text{ and } \|E^+ - w(s)\| \ge \frac{q_0}{2} \Rightarrow w_i'(s) \ge \beta. \tag{106}$$

Let $i \in \{1, \dots, N\}$. Since w is a stationary solution of (6),

$$\begin{aligned} \mathcal{N}_i(\mathbf{s}_{\eta,x_0})(x,t) = {}& \nu\eta\alpha_0 e^{-\nu t} w_i' - \nu\eta e^{-\nu t}\tilde{\mathfrak{p}}_i(w_i) - \nu\eta^2\alpha_0 e^{-2\nu t}\tilde{\mathfrak{p}}_i'(w_i) w_i' - \eta e^{-\nu t} A_i[\tilde{\mathfrak{p}}_i''(w_i)(w_i')^2 + \tilde{\mathfrak{p}}_i'(w_i) w_i''] \\ & - \eta e^{-\nu t} c\tilde{\mathfrak{p}}_i'(w_i) w_i' + f_i(w - \mathfrak{p}\eta e^{-\nu t}, w' - \eta e^{-\nu t} d\mathfrak{p}[w]w') - f_i(w, w'). \end{aligned}$$

By the Mean Value Theorem and the properties of $\mathfrak{p}$ and w,

$$f_i(w - \mathfrak{p}\eta e^{-\nu t}, w' - \eta e^{-\nu t} d\mathfrak{p}[w]w') - f_i(w, w') = \mathfrak{q}_i(x,t)\eta e^{-\nu t}, \tag{107}$$

where $\mathfrak{q}_i(x,t)$ is bounded independently of $x \in \mathbb{R}$ and $t \geq 0$. So

$$\begin{aligned} \mathcal{N}_i(\mathbf{s}_{\eta,x_0})(x,t) &= \eta e^{-\nu t}\left\{\mathfrak{q}_i(x,t) - \nu\tilde{\mathfrak{p}}_i(w_i) - A_i[\tilde{\mathfrak{p}}_i''(w_i)(w_i')^2 + \tilde{\mathfrak{p}}_i'(w_i)w_i''] - c\tilde{\mathfrak{p}}_i'(w_i)w_i'\right\} \\ &\quad + \nu\eta\alpha_0 e^{-\nu t} w_i'\{1 - \eta e^{-\nu t}\tilde{\mathfrak{p}}_i'(w_i)\}. \end{aligned} \tag{108}$$

Since $d\mathfrak{p}[\cdot]$ is uniformly bounded, there exists $\hat{\eta} \in (0, \eta_0]$ such that

$$\eta \in (0, \hat{\eta}] \Rightarrow 1 - \eta e^{-\nu t}|\tilde{\mathfrak{p}}_i'(\omega)| \geq \tfrac{1}{2} \quad \text{for each} \quad \omega \in \mathbb{R}. \tag{109}$$

(We need that $1 + \eta e^{-\nu t}\tilde{\mathfrak{p}}_i'(\omega) \geq \frac{1}{2}$ for the analysis of $\mathbf{S}_{\eta,x_1}$.) So since w_i' satisfies (106),

$$\begin{aligned} &\mathcal{N}_i(\mathbf{s}_{\eta,x_0})(x,t) \geq \\ &\quad \eta e^{-\nu t}\left\{\mathfrak{q}_i(x,t) - \nu\tilde{\mathfrak{p}}_i(w_i) - A_i[\tilde{\mathfrak{p}}_i''(w_i)(w_i')^2 + \tilde{\mathfrak{p}}_i'(w_i)w_i''] - c\tilde{\mathfrak{p}}_i'(w_i)w_i' + \tfrac{1}{2}\nu\beta\alpha_0\right\}. \end{aligned}$$

Whence we can choose $\alpha_0 > 0$, dependent on $\mathfrak{p}$ and w but independent of x and t, such that $\mathcal{N}_i(\mathbf{s}_{\eta,x_0})(x,t) \geq 0$. The result follows. $\square$

Theorem A.6 *Suppose that $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ satisfies* **(f1) - (f4)**. *Then there exists $\hat{\eta} > 0$ such that if $\phi \in \mathfrak{C}^1$ is such that there exists $\eta \in (0, \hat{\eta})$ with*

$$E^- - \eta\mathfrak{e}^- \leq \phi(x) \leq E^+ + \eta\mathfrak{e}^+ \quad \textit{for all} \quad x \in \mathbb{R}, \tag{110}$$

and

$$\limsup_{x\to\infty}|\phi_i(x) - E_i^+| \leq \eta\mathfrak{e}_i^+, \quad \limsup_{x\to-\infty}|\phi_i(x) - E_i^-| \leq \eta\mathfrak{e}_i^- \quad \textit{for each} \quad i \in \{1,\ldots,N\}, \tag{111}$$

then $\tau(\phi) = \infty$, and there exist $x_0(\phi), x_1(\phi) \in \mathbb{R}$ such that

$$\mathbf{s}_{\hat{\eta},x_0(\phi)}(x,t) \leq v^\phi(x,t) \leq \mathbf{S}_{\hat{\eta},x_1(\phi)}(x,t) \quad \textit{for all} \quad x \in \mathbb{R}, t \geq 0. \tag{112}$$

Proof. Let $\hat{\eta}$ be as in Theorem A.5, and let $\phi \in \mathfrak{C}^1$ satisfy (110, 111) for some $\eta \in (0, \hat{\eta})$. Now given $x_0, x_1 \in \mathbb{R}$,

$$\mathbf{s}_{\hat{\eta},x_0}(x,0) = w(x - x_0 + \hat{\eta}\alpha_0) - \hat{\eta}\mathfrak{p}(w(x - x_0 + \hat{\eta}\alpha_0)) \tag{113}$$

$$\mathbf{S}_{\hat{\eta},x_1}(x,0) = w(x + x_1 - \hat{\eta}\alpha_0) + \hat{\eta}\mathfrak{p}(w(x + x_1 - \hat{\eta}\alpha_0)) \tag{114}$$

for each $x \in \mathbb{R}$. Recall (103). So from (110), (111), (113), (114) and the fact that $\eta < \hat{\eta}$, it follows that there exist $x_0(\phi), x_1(\phi) \in \mathbb{R}$ such that

$$\mathbf{s}_{\hat{\eta},x_0(\phi)}(x,0) \leq \phi(x) \leq \mathbf{S}_{\hat{\eta},x_1(\phi)}(x,0) \quad \text{for all} \quad x \in \mathbb{R}. \tag{115}$$

This, together with Theorem A.5, allow application of Theorem A.2 to get that

$$\mathbf{s}_{\hat{\eta},x_0(\phi)}(x,t) \leq v^\phi(x,t) \leq \mathbf{S}_{\hat{\eta},x_1(\phi)}(x,t) \quad \text{for all} \quad x \in \mathbb{R}, 0 \leq t < \tau(\phi). \tag{116}$$

Whence condition (100) is satisfied. The result follows from Proposition A.4. $\square$

The wave-dependent sub- and super-solutions constructed above are valuable in analysing the stability of the wave w. The following is another, simple but useful, route to verification of condition (100).

Theorem A.7 *Suppose that* $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ *satisfies* **(f1)-(f4)**. *Then there exists* $\hat{\eta} > 0$ *such that if* $\phi \in \mathfrak{C}^1$ *is such that there exists* $\eta \in [0, \hat{\eta}]$ *such that*

$$E^- - \eta \mathfrak{e}^- \leq \phi(x) \leq E^+ + \eta \mathfrak{e}^+ \quad \textit{for all} \quad x \in \mathbb{R}, \tag{117}$$

then $\tau(\phi) = \infty$, *and*

$$E^- - \eta \mathfrak{e}^- \leq v^\phi(x,t) \leq E^+ + \eta \mathfrak{e}^+ \quad \textit{for all} \quad x \in \mathbb{R}, t \geq 0. \tag{118}$$

Proof. Let $\hat{\eta}$ be as in Theorem A.5 and let $\phi \in \mathfrak{C}^1$ satisfy (117) for some $\eta \in [0, \hat{\eta}]$. Then since $f(E^+, 0) = f(E^-, 0) = 0$ (by **(f3)**), it follows from (101) that $f(E^- - \eta \mathfrak{e}^-, 0) > 0, f(E^+ + \eta \mathfrak{e}^+, 0) < 0$. So with $\mathcal{N}$ as defined in (97), $\mathcal{N}(E^- - \eta \mathfrak{e}^-, 0) > 0$ and $\mathcal{N}(E^+ + \eta \mathfrak{e}^+, 0) < 0$. It then follows from Theorem A.2 that

$$E^- - \eta \mathfrak{e}^- \leq v^\phi(x,t) \leq E^+ + \eta \mathfrak{e}^+ \quad \text{for} \quad 0 \leq t \leq \tau(\phi). \tag{119}$$

Whence condition (100) is satisfied. The result follows from Proposition A.4. □

A priori bounds

The derivatives of v^ϕ can be estimated independently of the exact choice of ϕ satisfying (117), as follows.

Theorem A.8 *Let* $f, \hat{\eta}$ *be as in Theorem A.7 and let* $t_0 > 0$. *Then there exists* $K(t_0) > 0$ *such that if* $\phi \in \mathfrak{C}^1$ *satisfies (117) for some* $\eta \in [0, \hat{\eta}]$, *then for all* $t \geq t_0$,

$$\|v^\phi(\cdot, t)\|_{\mathfrak{C}^2} \leq K(t_0). \tag{120}$$

Proof. Since f satisfies **(f1), (f4)** and (118) holds, the single-equation analysis of [10, Chapter V, §3, p 437, Theorem 3.1] implies the existence of $K_1(t_0) > 0$ such that $\|v^\phi(\cdot, t)\|_{\mathfrak{C}^1} \leq K_1(t_0)$ for all $t \geq t_0$. This enables application of [10, Chapter VII, §5, p 586, Theorem 5.1] to obtain (120). □

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E.C.M.Crooks *
Department of Mathematical Sciences
University of Bath
Bath BA2 7AY
United Kingdom

* Current address : Balliol College, Oxford, OX1 3BJ, U.K., crooks@maths.ox.ac.uk